

# A nonlocal quasilinear multi-phase system with nonconstant specific heat and heat conductivity\*

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**Abstract.** In this paper, we prove the existence and global boundedness from above for a solution to an integrodifferential model for nonisothermal multi-phase transitions under nonhomogeneous third type boundary conditions. The system couples a quasilinear internal energy balance ruling the evolution of the absolute temperature with a vectorial integrodifferential inclusion governing the (vectorial) phase-parameter dynamics. The specific heat and the heat conductivity  $k$  are allowed to depend both on the order parameter  $\chi$  and on the absolute temperature  $\theta$  of the system, and the convex component of the free energy may or may not be singular. Uniqueness and continuous data dependence are also proved under additional assumptions.

**Key words:** Phase transitions, nonlocal models, quasilinear integrodifferential vectorial equation

**AMS (MOS) subject classification:** 35K51, 35K59, 35K65, 45K05, 80A22

## 1 Introduction

In this paper we consider a nonisothermal multi-phase transition process occurring in a bounded container  $\Omega \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ , with Lipschitz boundary  $\partial\Omega$ . The state variables

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\***Acknowledgment.** The authors gratefully acknowledge financial support by the MIUR-PRIN Grant 2008ZKHAHN “Phase transitions, hysteresis and multiscaling”, the GAČR Grant P201/10/2315, the program SMART-MATH of CNR/AVČR, the MIUR-PRIN Grant 20089PWTPS “Mathematical Analysis for inverse problems towards applications”, the DFG Research Center MATHEON in Berlin, and the IMATI of CNR in Pavia.

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describing the evolution of the system are the absolute temperature  $\theta > 0$  and the vectorial order parameter  $\chi \in \mathbb{R}^d$ ,  $d \in \mathbb{N}$ . Following the idea that was already described in the pioneering papers [26] and [4], but which has been only recently analyzed in a more systematic way (cf., e.g., [1]–[2], [5]–[6], [8]–[21], [24]), we take into account long range interactions between particles. Then the model equations resulting from the energy and entropy balance relations have the form

$$\begin{aligned} (1.1) \quad & (e(\theta, \chi))_t + (\lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \operatorname{div}(k(\theta, \chi)\nabla\theta) = 0 \quad \text{in } Q_\infty := \Omega \times (0, +\infty), \\ (1.2) \quad & \mu(\theta)\chi_t + \lambda'(\chi) + b[\chi] + (\beta + \theta)\partial\varphi(\chi) + \theta\sigma'(\chi) + e_\chi(\theta, \chi) - \theta s_\chi(\theta, \chi) \ni 0 \quad \text{in } Q_\infty, \\ (1.3) \quad & k(\theta, \chi)\nabla\theta \cdot \mathbf{n} + \gamma(\theta - \theta_\Gamma) = 0 \quad \text{on } \Sigma_\infty := \partial\Omega \times (0, \infty), \\ (1.4) \quad & \theta(\cdot, 0) = \theta_0, \quad \chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega, \end{aligned}$$

where  $\mathbf{n}$  denotes the outward normal vector to  $\partial\Omega$ , and (1.2) has to be understood as an inclusion in  $\mathbb{R}^d$ , where  $\partial\varphi$  is a possibly multivalued subdifferential of a general proper, convex, and lower semicontinuous function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ . The physical meaning of the functions  $e$ ,  $s$ ,  $\lambda$ ,  $\sigma$  and of the positive constant  $\beta$  is explained in (1.16)–(1.19), while  $b[\chi]$  (whose explicit form will be given) represents the nonlocal operator acting on  $\chi$ . With an abuse of notation we have used the symbols  $\lambda'$ ,  $\sigma'$ ,  $e_\chi$ ,  $s_\chi$  for gradient vectors in  $\mathbb{R}^d$  and omitted the scalar product symbol between  $\mathbb{R}^d$  vectors (like  $b[\chi]$  and  $\chi_t$  in (1.1)) in order not to overburden the presentation. The function  $\mu$  in (1.2) represents the (bounded away from 0) mobility of the system, while  $\gamma$  denotes the heat transfer coefficient through the boundary  $\partial\Omega$ . The external temperature  $\theta_\Gamma$  is a sufficiently regular boundary datum on  $\Sigma_\infty$ , and  $\theta_0$ ,  $\chi_0$  are supposed to be two given initial configurations.

The main novelty here is to consider a multi-phase nonlocal phase field system in the case when the specific heat  $c_V(\theta, \chi) = \partial_\theta e(\theta, \chi)$  and the heat conductivity  $k(\theta, \chi)$  are not constant and depend on both the variables  $\theta$  and  $\chi$ . Suitable regularity and growth conditions will be specified in the following section.

Let us only note that many typical expressions for  $c_V$  in a two-phase system (i.e. in case  $d = 1$ ) can be included in our analysis. In the solid-liquid system mentioned above, for example, we may have different values  $c_V^0(\theta)$  in the solid and  $c_V^1(\theta)$  in the liquid phase, hence, we may define  $c_V(\theta, \chi) = c_V^0(\theta) + \chi(c_V^1(\theta) - c_V^0(\theta))$  (cf. [25, Section IV.4]). The value of  $\chi$  can be kept between 0 and 1 by setting  $\varphi = I_{[0,1]}$  (the indicator function of  $[0, 1]$ ). The physically meaningful case in which the behaviour of  $c_V^0$  and  $c_V^1$  are powers of  $\theta$  ( $\sim \theta^\alpha$ ,  $\alpha \geq 1$ ) near zero and bounded functions for large  $\theta$ 's can be covered by our analysis. Regarding the heat conductivity  $k$ , typical expressions of the type  $k(\theta, \chi) = K_1(\theta)\chi + K_2(\theta)(1 - \chi)$ , in case of a two phase system with  $\chi \in [0, 1]$ , for quite general functions  $K_1$  and  $K_2$ , are also allowed here.

The main goal of this paper is to study the global existence of solutions to system (1.1)–(1.4), coupling a suitable variational formulation of the semilinear parabolic partial differential equation (1.1) for  $\theta$  to the integrodifferential inclusion (1.2) for  $\chi$ . We also prove some uniform in time upper bound for the absolute temperature of the system (see Theorem 2.2 below). Uniqueness of solutions is obtained under additional assumptions, in particular, in case that the heat conductivity  $k$  in (1.1) depends only on  $\theta$  and not on  $\chi$ .

Before entering into the mathematical discussion of the problem, let us give a brief

derivation of the system (1.1)–(1.4), emphasizing, in particular, the differences between local and nonlocal models.

We assume here that the multi-phase transition process can be completely described by the evolution of the state variables  $\theta(x, t) > 0$ , which represents the absolute temperature of the system, and the order parameter  $\chi(x, t)$ , which here is a vector in  $\mathbb{R}^d$ . We fix some constant reference temperature  $\theta_c$ , which will be assumed to be equal to 1, for simplicity.

Inspired by the nonlocal Cahn-Hilliard model studied by Gajewski in [9], we consider the following nonlocal specific free energy

$$F[\theta, \chi] = f_0(\theta, \chi) + B[\chi],$$

where  $B$  is a potential that accounts for long range interaction between particles. More specifically, given a bounded, symmetric kernel  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}$  and an even smooth function  $G : \mathbb{R}^d \rightarrow \mathbb{R}$ , we choose

$$(1.5) \quad B[\chi](x, t) := \int_{\Omega} \kappa(x, y) G(\chi(x, t) - \chi(y, t)) \, dy.$$

Note that the local potential  $(\nu/2)|\nabla\chi|^2$  used often in the literature, see [25] and the references therein, can be obtained as a formal limit as  $n \rightarrow \infty$  from the nonlocal one with the choice  $G(\eta) = |\eta|^2/2$ ,  $\kappa(x, y) = n^{N+2}\tilde{\kappa}(|n(x-y)|^2)$ , where  $\tilde{\kappa}$  is a nonnegative function with support in  $[0, 1]$  and  $\nu = 1/N \int_{\mathbb{R}^N} \tilde{\kappa}(|z|^2) |z|^2 \, dz$ . This follows from the formula

$$\begin{aligned} \int_{\Omega} n^{N+2}\tilde{\kappa}(|n(x-y)|^2) |\chi(x) - \chi(y)|^2 \, dy &= \int_{\Omega_n(x)} \tilde{\kappa}(|z|^2) \left| \frac{\chi\left(x + \frac{z}{n}\right) - \chi(x)}{\frac{1}{n}} \right|^2 \, dz \\ &\xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^N} \tilde{\kappa}(|z|^2) \langle \nabla\chi(x), z \rangle^2 \, dz = \nu |\nabla\chi(x)|^2 \end{aligned}$$

for a sufficiently regular  $\chi$ , where we denote  $\Omega_n(x) = n(\Omega - x)$ . We have used the identity  $\int_{\mathbb{R}^N} \tilde{\kappa}(|z|^2) \langle v, z \rangle^2 \, dz = 1/N \int_{\mathbb{R}^N} \tilde{\kappa}(|z|^2) |z|^2 \, dz$  for every unit vector  $v \in \mathbb{R}^N$  (cf. the Introduction of [19] for further details on this topic).

Let  $E$  and  $S$  be the total energy and entropy densities, respectively. The process is governed by the internal energy and entropy balance relations over an arbitrary control volume  $\Omega' \subset \Omega$ ,

$$(1.6) \quad \frac{d}{dt} \int_{\Omega'} E(\theta, \chi) \, dx + \int_{\partial\Omega'} \langle \mathbf{q}, \mathbf{n} \rangle \, ds(x) = \Psi(\Omega'),$$

$$(1.7) \quad \frac{d}{dt} \int_{\Omega'} S(\theta, \chi) \, dx + \int_{\partial\Omega'} \left\langle \frac{\mathbf{q}}{\theta}, \mathbf{n} \right\rangle \, ds(x) \geq 0,$$

where  $\mathbf{q}$  is the heat flux vector,  $\mathbf{n}$  is the unit outward normal to  $\partial\Omega'$ , and  $\Psi(\Omega')$  is the energy exchange through the boundary of  $\Omega'$  due to the nonlocal interactions. Since  $B[\chi]$  is a potential field, it does not contribute to the entropy production in the Clausius-Duhem inequality (1.7).

The local form of the entropy balance reads

$$\theta S_t(\theta, \chi) + \operatorname{div} \mathbf{q} - \frac{\langle \mathbf{q}, \nabla \theta \rangle}{\theta} \geq 0,$$

and it has to be understood in the regularity context of Theorem 2.2 below. This is certainly satisfied if

$$\begin{aligned} \langle \mathbf{q}, \nabla \theta \rangle &\leq 0, \\ \theta S_t(\theta, \chi) + \operatorname{div} \mathbf{q} &\geq 0. \end{aligned}$$

Assuming  $\theta > 0$  and a suitable regularity with respect to time (this will have to be justified in the next sections), we obtain from (1.6) that

$$(1.8) \quad \int_{\Omega'} (E_t - \theta S_t) \, dx \leq \Psi(\Omega').$$

Differentiating the identities  $F = E - \theta S = f_0 + B[\chi]$  with respect to  $t$ , we obtain

$$(1.9) \quad F_t = E_t - \theta S_t - \theta_t S = \partial_\theta f_0 \theta_t + \partial_\chi f_0 \chi_t + B[\chi]_t,$$

where  $\partial_\chi f_0$  stands for an element of Clarke's partial subdifferential of  $f_0$  with respect to  $\chi \in \mathbb{R}^d$ , and  $\partial_\theta f_0$  is the partial derivative of  $f_0$  with respect to  $\theta \in \mathbb{R}$ . Consequently,

$$(1.10) \quad S = -\partial_\theta f_0 = s_0, \quad E = e_0 + B[\chi], \quad f_0 = e_0 - \theta s_0,$$

and inequality (1.8) reads

$$(1.11) \quad \int_{\Omega'} (\partial_\chi f_0 \chi_t + B[\chi]_t) \, dx \leq \Psi(\Omega').$$

The nonlocal interaction takes place only inside the domain  $\Omega$ , hence  $\Psi(\Omega) = 0$ . A canonical way to satisfy these conditions independently of the evolution of  $\chi$  consists in choosing the order parameter dynamics in the form

$$(1.12) \quad \mu(\theta) \chi_t \in -D_\chi \mathcal{F}[\theta, \chi]$$

with a factor  $\mu(\theta) > 0$ , where we denote

$$\mathcal{F}[\theta, \chi] = \int_{\Omega} F[\theta, \chi] \, dx$$

and  $D_\chi \mathcal{F}$  stands for the Clarke subdifferential of  $\mathcal{F}$  with respect to the variable  $\chi \in L^2(\Omega; \mathbb{R}^d)$ . The inclusion sign in (1.12) accounts for the fact that  $f_0(\theta, \chi)$  includes terms that are possibly not Fréchet differentiable. Condition (1.12) is based on the assumption that the system tends to move towards local minima of the free energy with a speed proportional to  $1/\mu(\theta)$ . Denoting

$$(1.13) \quad b[\chi](x, t) := 2 \int_{\Omega} \kappa(x, y) G'(\chi(x, t) - \chi(y, t)) \, dy,$$

where again, with an abuse of notation,  $G'$  stands for the  $d$ -component vector  $\nabla G$ , we see that the inequality (1.11) holds without prescribing any relationship between  $\mu(\theta)$  and  $B[\chi]$ , provided that we choose  $\Psi(\Omega')$  in (1.6) as

$$(1.14) \quad \Psi(\Omega') = \int_{\Omega'} (-b[\chi]\chi_t + B[\chi]_t) \, dx.$$

The differential form of the energy balance (1.6) then reads

$$(1.15) \quad E_t + \operatorname{div} \mathbf{q} = -b[\chi]\chi_t + B[\chi]_t.$$

The specific heat  $c_V(\theta, \chi)$  is the only thermodynamic state function, which can be identified from the measurements, while the local internal energy and entropy densities are computed from the formulas

$$(1.16) \quad e_0(\theta, \chi) = e_0(0, \chi) + e(\theta, \chi), \quad e(\theta, \chi) = \int_0^\theta c_V(\tau, \chi) \, d\tau,$$

$$(1.17) \quad s_0(\theta, \chi) = s_0(0, \chi) + s(\theta, \chi), \quad s(\theta, \chi) = \int_0^\theta \frac{c_V(\tau, \chi)}{\tau} \, d\tau,$$

where  $e_0(0, \chi), s_0(0, \chi)$  are in fact “integration constants”, which we choose as

$$(1.18) \quad e_0(0, \chi) = \lambda(\chi) + \beta\varphi(\chi),$$

$$(1.19) \quad s_0(0, \chi) = -\sigma(\chi) - \varphi(\chi),$$

where  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is proper, convex, and lower semicontinuous, the functions  $\lambda$  and  $\sigma$  are sufficiently regular on  $\mathcal{D}(\varphi)$ , and the parameter  $\beta$  is a positive constant.

Then, the free energy functional  $F$  has the form

$$(1.20) \quad F[\theta, \chi] = f(\theta, \chi) + \lambda(\chi) + B[\chi] + (\beta + \theta)\varphi(\chi) + \theta\sigma(\chi),$$

where  $f(\theta, \chi) = e(\theta, \chi) - \theta s(\theta, \chi)$ .

Using (1.20), we rewrite the phase dynamics (1.12) as

$$(1.21) \quad \mu(\theta)\chi_t + \lambda'(\chi) + b[\chi] + (\beta + \theta)\partial\varphi(\chi) + \theta\sigma'(\chi) + e_\chi(\theta, \chi) - \theta s_\chi(\theta, \chi) \ni 0,$$

while the internal energy balance (1.15) can be reformulated as

$$(1.22) \quad (e(\theta, \chi))_t + (\lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \operatorname{div}(k(\theta, \chi)\nabla\theta) = 0.$$

We now show that in the energy conserved case, that is, if we assume no-flux boundary conditions ( $\gamma = 0$  in (1.3)), the phase transition model with a nonlocal interaction potential is compatible with the Öttinger-Grmela GENERIC formalism (abbreviation for “General Equation for the Non-Equilibrium Reversible-Irreversible Coupling”), see [16].

Set

$$(1.23) \quad \mathcal{E}[\theta, \chi](t) = \int_{\Omega} E(\theta, \chi)(x, t) \, dx,$$

$$(1.24) \quad \mathcal{S}[\theta, \chi](t) = \int_{\Omega} S(\theta, \chi)(x, t) \, dx,$$

$$(1.25) \quad \mathcal{B}[\chi](t) = \int_{\Omega} B[\chi](x, t) \, dx.$$

We show that there exists a symmetric positive semidefinite matrix  $\mathbf{M}[\theta, \chi]$  such that

$$(1.26) \quad \mathbf{M}[\theta, \chi] \begin{pmatrix} D_{\theta} \mathcal{E}[\theta, \chi] \\ D_{\chi} \mathcal{E}[\theta, \chi] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and such that the system (1.21)–(1.22) has the form

$$(1.27) \quad \frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \chi \end{pmatrix} = \mathbf{M}[\theta, \chi] \begin{pmatrix} D_{\theta} \mathcal{S}[\theta, \chi] \\ D_{\chi} \mathcal{S}[\theta, \chi] \end{pmatrix}.$$

It suffices to choose (we omit the arguments of the state functions for simplicity)

$$(1.28) \quad \mathbf{M}[\theta, \chi] = \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix},$$

where  $M_0$  is the differential operator

$$(1.29) \quad M_0[y] = -\frac{1}{c_V} \operatorname{div} \left( \theta^2 k(\theta, \chi) \nabla \frac{y}{c_V} \right),$$

with homogeneous Neumann boundary condition, and  $m_{ij}$  are scalars given by the formulas

$$(1.30) \quad m_{11} = \frac{\theta}{\mu(\theta) c_V^2} (D_{\chi} \mathcal{E})^2,$$

$$(1.31) \quad m_{12} = -\frac{\theta}{\mu(\theta) c_V} D_{\chi} \mathcal{E},$$

$$(1.32) \quad m_{22} = \frac{\theta}{\mu(\theta)}.$$

Note that  $m_{12}^2 = m_{11} m_{22}$  and  $m_{11} \geq 0, m_{22} \geq 0$ ; hence,  $\mathbf{M}[\theta, \chi]$  is positive semidefinite. Furthermore, we have

$$(1.33) \quad \begin{pmatrix} D_{\theta} \mathcal{E} \\ D_{\chi} \mathcal{E} \end{pmatrix} = \begin{pmatrix} c_V \\ D_{\chi} \mathcal{E} \end{pmatrix} = \begin{pmatrix} c_V \\ \frac{\partial}{\partial \chi} e_0 + D_{\chi} \mathcal{B} \end{pmatrix}, \quad \begin{pmatrix} D_{\theta} \mathcal{S} \\ D_{\chi} \mathcal{S} \end{pmatrix} = \begin{pmatrix} c_V / \theta \\ \frac{\partial \mathcal{S}}{\partial \chi} \end{pmatrix}.$$

We easily check that (1.26) holds, and (1.27) has the form

$$(1.34) \quad \begin{pmatrix} \theta_t \\ \chi_t \end{pmatrix} = \begin{pmatrix} \operatorname{div} (k(\theta, \chi) \nabla \theta) / c_V \\ 0 \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} c_V / \theta \\ \frac{\partial \mathcal{S}}{\partial \chi} \end{pmatrix}.$$

In component form, we have

$$(1.35) \quad \theta_t = \frac{1}{c_V} \left( \operatorname{div} (k(\theta, \chi) \nabla \theta) + \frac{1}{\mu(\theta)} D_\chi \mathcal{E} D_\chi \mathcal{F} \right),$$

$$(1.36) \quad \chi_t = -\frac{1}{\mu(\theta)} D_\chi \mathcal{F}.$$

To see that (1.35)–(1.36) coincides with (1.21)–(1.22), it suffices to take into account the formula

$$\frac{\partial}{\partial t} e_0 = c_V \theta_t + \frac{\partial}{\partial \chi} e_0 \chi_t = \operatorname{div} (k(\theta, \chi) \nabla \theta) + \frac{1}{\mu(\theta)} D_\chi \mathcal{B} D_\chi \mathcal{F} = \operatorname{div} (k(\theta, \chi) \nabla \theta) - \chi_t D_\chi \mathcal{B}.$$

This shows that the model is compatible both with the standard principles of thermodynamics and the generalized thermodynamic formalism introduced in [16]. Note that Mielke [22] recently elaborated the GENERIC approach for the phenomenology of thermoelastic dissipative materials.

We prove an existence result for a suitable variational formulation of system (1.1)–(1.4). Using a Moser technique, we also show that the temperature variable  $\theta$  is globally bounded from above. The uniqueness result holds true for particular classes of potentials  $\varphi$  provided that the heat conductivity  $k$  in (1.1) does not depend on  $\chi$ ,  $\gamma = 0$  in (1.3), and  $c_V$  and  $\mu$  satisfy a suitable growth condition around 0.

The paper is organized as follows. In Section 2, we state our assumptions on the data and our main results; in particular, global existence for a suitable variational formulation of (1.1)–(1.4). In Section 3, we prove some auxiliary results related to the Lipschitz continuity of solution operators to differential inclusions. The proof will be developed as follows: the problem is approximated by partial time discretization, regularization and a cut-off procedure (cf. Subsection 4.1). Suitable a priori estimates (cf. Subsection 4.2) allow us to pass to the limit with respect to the time step and regularization parameters, while the cut-off is removed by proving an upper bound on the absolute temperature (which is independent of the truncation parameter) by means of Moser techniques (cf. Subsection 4.3). The uniqueness result is proved in Section 5.

## 2 Main results

In this section, we state our main results on solvability conditions for the system (1.1)–(1.4). We start by introducing a suitable variational formulation; to this end, we consider a bounded and Lipschitz domain  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , and for  $t \in (0, \infty]$  we denote by  $Q_t = \Omega \times (0, t)$  the open space-time cylinder and by  $\Sigma_t$  its lateral boundary  $\partial\Omega \times (0, t)$ . We use, for the sake of simplicity, the same symbol  $H$  for both  $L^2(\Omega)$  and  $L^2(\Omega; \mathbb{R}^N)$ , while for an arbitrary integer  $d$ ,  $\mathbf{H}$  denotes the space  $L^2(\Omega; \mathbb{R}^d)$ .  $H$  and  $\mathbf{H}$  are both endowed with the standard scalar product which we denote by  $(\cdot, \cdot)$ . The symbol  $V$  stands for the space  $H^1(\Omega)$ , and  $V'$  for its dual space, while the symbol  $\mathbf{V}$  denotes the space  $H^1(\Omega; \mathbb{R}^d)$ ,  $\langle \cdot, \cdot \rangle$  being the duality  $V' - V$  and  $\mathbf{V}' - \mathbf{V}$ . Then, the following dense and continuous embeddings, where we

identify  $H$  (and  $\mathbf{H}$ ) with its dual space  $H'$  (and  $\mathbf{H}'$ ), hold true:  $V \hookrightarrow H \equiv H' \hookrightarrow V'$ , and  $\mathbf{V} \hookrightarrow \mathbf{H} \equiv \mathbf{H}' \hookrightarrow \mathbf{V}'$ . Finally, we rewrite the system (1.1)–(1.4) in the following variational formulation:

$$(2.1) \quad \langle \partial_t(e(\theta, \chi)), z \rangle + \int_{\Omega} k(\theta, \chi) \nabla \theta \cdot \nabla z \, dx + \int_{\partial\Omega} \gamma(\theta - \theta_{\Gamma}) z \, dA \\ = - \int_{\Omega} (\lambda'(\chi) \partial_t \chi + \beta(\varphi(\chi))_t + b[\chi] \chi_t) z \, dx \quad \forall z \in V, \quad \text{a.e. in } (0, \infty),$$

$$(2.2) \quad \mu(\theta) \chi_t + \lambda'(\chi) + \theta \sigma'(\chi) + (\beta + \theta) \partial \varphi(\chi) + b[\chi] \\ + e_{\chi}(\theta, \chi) - \theta s_{\chi}(\theta, \chi) \ni 0 \quad \text{a.e. in } Q_{\infty},$$

where (2.2) has to be understood as an inclusion in  $\mathbb{R}^d$  with  $b[\chi]$  defined by (1.13), and  $e$  and  $s$  are defined in (1.16)–(1.17). Letting (cf. (1.4))  $u_0 := e(\theta_0, \chi_0)$ , we prescribe the initial conditions

$$(2.3) \quad e(\theta, \chi)(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega,$$

and suppose that the data fulfil the following assumptions.

**Hypothesis 2.1.** (Existence) *Let us fix positive constants  $C_{\sigma}$ ,  $C_{\lambda}$ ,  $k_0$ ,  $k_1$ ,  $\underline{c}$ ,  $\bar{c}$ ,  $c_1$ ,  $\beta$ ,  $C_0$ , and assume that*

- (i)  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex, and lower semicontinuous function,  $\mathcal{D}(\varphi)$  is its domain;
- (ii)  $\sigma, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi))$ ,  $|\sigma'(r)| \leq C_{\sigma}$ ,  $|\lambda'(r)| \leq C_{\lambda}$  for all  $r \in \mathcal{D}(\varphi)$ ;
- (iii)  $\kappa \in W^{1,\infty}(\Omega \times \Omega)$ ,  $\kappa(x, y) = \kappa(y, x)$  a.e. in  $\Omega \times \Omega$ ,  $G \in W^{2,\infty}(\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$ ,  $G(z) = G(-z)$  for all  $z \in (\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$ ;
- (iv)  $k : \mathbb{R} \times \mathcal{D}(\varphi) \rightarrow (0 + \infty)$  is a locally Lipschitz continuous function such that  $0 < k_0 \leq k(v, w) \leq k_1$  for all  $v \in \mathbb{R}$  and  $w \in \mathcal{D}(\varphi)$ ;
- (v) The function  $\mu$  maps  $[0, \infty)$  in  $(0, \infty)$  and the function  $\theta \mapsto \frac{1+\theta}{\mu(\theta)}$  is bounded and Lipschitz continuous on  $[0, \infty)$  with Lipschitz constant  $L_{\mu}$ ;
- (vi)  $c_V : [0, +\infty) \times \mathcal{D}(\varphi) \rightarrow [0, +\infty)$  is a continuous function satisfying

$$(2.4) \quad c_V(0, \chi) = 0, \quad 0 < c_V(\theta, \chi) \leq \bar{c} \quad \forall \theta \in (0, +\infty), \quad \forall \chi \in \mathcal{D}(\varphi);$$

$$(2.5) \quad \underline{c} \leq c_V(\theta, \chi) \quad \forall (\theta, \chi) \in [1, +\infty) \times \mathcal{D}(\varphi);$$

$$(2.6) \quad \text{the function } \theta \mapsto \frac{c_V(\theta, \chi)}{\theta} \text{ is integrable in } (0, 1) \text{ for all } \chi \in \mathcal{D}(\varphi).$$

Moreover, for all  $(\theta, \chi) \in [0, +\infty) \times \mathcal{D}(\varphi)$  there exists the gradient  $(c_V)_{\chi}(\theta, \chi)$ , and it holds

$$(2.7) \quad |(c_V)_{\chi}(\theta, \chi)| \leq c_1 c_V(\theta, \chi), \quad |(c_V)_{\chi}(\theta, \chi_1) - (c_V)_{\chi}(\theta, \chi_2)| \leq c_1 |\chi_1 - \chi_2| \\ \text{for all } \theta \in [0, +\infty), \quad \chi, \chi_1, \chi_2 \in \mathcal{D}(\varphi).$$



Let  $e$  and  $s$  be defined by formulas (1.16)–(1.17) and suppose that

$$(2.8) \quad 0 < s(1, \chi) \leq c_1 \quad \forall \chi \in \mathcal{D}(\varphi);$$

$$(2.9) \quad |s_\chi(\theta_1, \chi_1) - s_\chi(\theta_2, \chi_2)| \leq c_1 (|\theta_1 - \theta_2| + |\chi_1 - \chi_2|) \\ \text{for all } \theta_1, \theta_2 \in [0, +\infty), \chi_1, \chi_2 \in \mathcal{D}(\varphi).$$

(vii)  $\chi_0 \in \mathbf{V} \cap L^\infty(\Omega)^d$ . Moreover, for any  $C > 0$  set

$$\mathcal{D}_C(\varphi) = \{\chi \in \mathcal{D}(\varphi) : \exists \xi \in \partial\varphi(\chi) : |\xi| \leq C\},$$

and assume that  $\chi_0(x) \in \mathcal{D}_{C_0}(\varphi)$  a.e. in  $\Omega$ ;

(viii)  $\theta_0, u_0 \in L^\infty(\Omega)$  fulfil  $u_0 = e(\theta_0, \chi_0)$  and  $\theta_0(x) > 0$  a.e. in  $\Omega$ ;

(ix)  $\gamma \in L^\infty(\partial\Omega)$  is a nonnegative function;

(x)  $\theta_\Gamma \in L^\infty(\Sigma_\infty)$  is such that  $\theta_\Gamma(x, t) > 0$  a.e. and  $\log(\theta_\Gamma) \in L^1(\Sigma_\infty)$ ;

(xi) Let  $\zeta \in (W_{loc}^{1,1}(0, \infty))^d$  be the solution to the differential inclusion

$$(2.10) \quad \alpha(t)\zeta_t + \partial\varphi(\zeta) \ni g(t) \quad \text{a.e.},$$

with the initial condition  $\zeta(0) = \zeta_0$ ,  $\zeta_0 \in \mathbb{R}^d$ , and given data  $g \in (L^\infty(0, \infty))^d$  and  $\alpha \in L_{loc}^\infty(0, \infty)$  such that  $0 < \alpha_0 \leq \alpha(t)$  a.e. We assume that there exists a positive constant  $D > 0$  such that for all  $C > 0$  such that  $|g(t)| \leq C$ , and  $\zeta_0 \in \mathcal{D}_C(\varphi)$ , we have

$$(2.11) \quad |(g - \alpha\zeta_t)(t)| \leq DC \quad \text{a.e.}$$

We are now in position to state the existence theorem.

**Theorem 2.2.** (Existence) Let Hypothesis 2.1 hold. Then there exists at least one pair  $(\theta, \chi)$  that solves system (2.1)–(2.3) and such that

$$(2.12) \quad \theta \in L^\infty(Q_\infty) \cap L^2(0, \infty; V), \quad (e(\theta, \chi))_t \in L_{loc}^2(0, \infty; V'),$$

$$(2.13) \quad \theta(x, t) > 0 \quad \text{a.e. in } Q_\infty,$$

$$(2.14) \quad \chi \in L_{loc}^\infty(Q_\infty)^d \cap L_{loc}^\infty(0, \infty; \mathbf{V}), \quad \chi_t \in L^\infty(Q_\infty)^d; \\ \exists C > 0 : \chi(x, t) \in \mathcal{D}_C(\varphi) \quad \text{a.e. in } Q_\infty.$$

Moreover, there exists a positive constant  $\bar{\theta}$  independent of  $t$  such that the following uniform upper bound hold:

$$(2.15) \quad \theta(x, t) < \bar{\theta} \quad \text{for a.e. } (x, t) \in Q_\infty.$$

**Hypothesis 2.3.** (Uniqueness) Assume that Hypothesis 2.1 is satisfied and suppose moreover that

(i)  $k(\theta, \chi) = \bar{k}(\theta)$  for all  $\theta \in \mathbb{R}$  and  $\chi \in \mathbb{R}^d$ ;

(ii) Fix  $T \in (0, \infty)$  and suppose that there exists a positive constant  $R$ , depending only on  $C$ ,  $\alpha_0$ , and  $T$  such that the solutions  $\zeta_1, \zeta_2 \in W^{1,\infty}(0, T)$  to (2.10) associated with data  $\zeta_{01}, \zeta_{02} \in \mathcal{D}_C(\varphi)$ ,  $\alpha_1, \alpha_2 \in L^\infty(0, T)$ , and with  $g_1, g_2 \in L^\infty(0, T)$  complying with the constraint

$$(2.16) \quad |g_i(t)| \leq C, \quad i = 1, 2, \quad \text{a.e. in } (0, T),$$

satisfy for all  $t \in (0, T)$  the inequality

$$(2.17) \quad \int_0^t |\dot{\zeta}_1 - \dot{\zeta}_2|(\tau) \, d\tau + |\zeta_1 - \zeta_2|(t) \leq R \left( |\zeta_{01} - \zeta_{02}| + \int_0^t \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right|(\tau) + |g_1 - g_2|(\tau) \right) \, d\tau \right);$$

(iii) Define  $\tilde{c}(\theta) := \min\{c_V(v, \chi) : \chi \in \mathcal{D}(\varphi), v \geq \theta\}$  and assume that  $\int_0^1 \frac{\tilde{c}(v)\mu(v)}{v^2} \, dv = +\infty$ ;

(iv) The function  $v \mapsto v^2/\mu(v)$  is nondecreasing in  $(0, +\infty)$ ;

(v) There exists  $\theta_* > 0$  such that  $\theta_0(x) \geq \theta_*$  for a.e.  $x \in \Omega$ , and  $\gamma \equiv 0$  on  $\partial\Omega$ ;

(vi) Assume that  $\tilde{c}(\theta) > 0$  for every  $\theta \in (0, \infty)$ .

**Remark 2.4.** Hypothesis 2.1 allows for physically meaningful choices of  $c_V$ . We can choose, for example, a  $(d+1)$ -component model with  $K = \{\chi_i \geq 0, \sum_{i=1}^d \chi_i \leq 1\}$ ,  $\chi_0 = 1 - \sum_{i=1}^d \chi_i$ ,  $\varphi = I_K$  (indicator function of the set  $K$ ), and  $c_V(\theta, \chi) = \sum_{i=1}^d c_i(\theta)\chi_i$ , where  $c_i(\theta)$  behave asymptotically at 0 and  $\infty$  like  $\frac{\theta^\alpha}{1 + \theta^\alpha}$ . Further examples of potentials  $\varphi$  complying with Hypothesis 2.1(xi) and Hypothesis 2.3(ii) will be given in the following Section 3.

We state then our last result regarding uniqueness and continuous data dependence for (2.1)–(2.3).

**Theorem 2.5.** (Uniqueness) Suppose that Hypothesis 2.3 is satisfied. Let  $T \in (0, \infty)$  be fixed. Then, there exists a positive constant  $\underline{\theta}(T)$  such that

$$(2.18) \quad \theta(x, t) \geq \underline{\theta}(T) \quad \text{for a.e. } (x, t) \in Q_T.$$

Moreover, if  $(\theta_1, \chi_1), (\theta_2, \chi_2)$  are two solutions to (2.1)–(2.3) in the sense of Theorem 2.2 associated with initial data  $\theta_{01}, \chi_{01}$  and  $\theta_{02}, \chi_{02}$ , respectively, and  $\hat{\theta} = \theta_1 - \theta_2$ ,  $\hat{\chi} = \chi_1 - \chi_2$ ,  $\hat{\chi}_0 = \chi_{01} - \chi_{02}$ ,  $\hat{\theta}_0 = \theta_{01} - \theta_{02}$ , then, there exists a constant  $C_T > 0$  such that

$$(2.19) \quad \int_0^T \int_\Omega |\hat{\theta}(x, t)|^2 \, dx \, dt + \max_{t \in [0, T]} \int_\Omega |\hat{\chi}(x, t)|^2 \, dx \leq C_T \left( |\hat{\theta}_0|_H^2 + |\hat{\chi}_0|_{\mathbf{H}}^2 \right).$$

Finally, beside Hypothesis 2.3, assume that

$$(2.20) \quad \theta_0 \in V.$$

Then, the  $\theta$ -component of the solution  $(\theta, \chi)$  to (2.1)–(2.3) has the further regularity

$$(2.21) \quad \theta \in L^\infty(0, T; V), \quad \theta_t \in L^2(0, T; H).$$

### 3 A differential inclusion

This section is devoted to the description of some properties of solutions to general differential inclusions of the form (2.10), which are used in the proof of Theorems 2.2, 2.5.

First we provide some examples of functions  $\varphi$  satisfying Hypothesis 2.1(xi), and we prove some further properties for space and time dependent differential inclusions that follow exactly from this assumption. Finally, we give examples of functions  $\varphi$  satisfying Hypothesis 2.3(ii).

#### Examples of functions complying with Hypothesis 2.1(xi).

**Proposition 3.1.** *The function  $\varphi$  introduced in Hypothesis 2.1(i) also satisfies Hypothesis 2.1(xi) in each of the following cases:*

- (a) if  $d = 1$ ;
- (b) if  $\varphi$  is the indicator function  $I_K$  associated with a closed, and convex set  $K \subset \mathbb{R}^d$ . In this case one has  $D = 1$ ;
- (c) if  $\varphi(x) = f(M_K(x))$ , where  $f : [0, f_0) \rightarrow [0, +\infty)$  is an increasing and convex  $C^1$  function such that  $f(0) = f'(0) = 0$ ,  $f_0 > 0$ , and  $M_K$  is the Minkowski functional of  $K$ , a closed, convex set in  $\mathbb{R}^d$  such that  $B_r(0) \subset K \subset B_R(0)$ , defined by the formula  $M_K(x) = \inf \{s > 0 ; \frac{1}{s}x \in K\}$ , for  $x \in \mathbb{R}^d$ . Then  $D = R/r$ .

The proof of point (a) follows directly from [18, Prop. 3.4], (b) is obvious. Let us prove the point (c). In order to do that, we first need to prove the following auxiliary result.

**Proposition 3.2.** *Let  $\varphi(x)$  be as in Proposition 3.1(c). Then for every  $C > 0$  there exists  $C_1 > 0$  such that for every  $x \in \mathcal{D}(\varphi)$  we have the following implications*

$$(3.1) \quad \varphi(x) \leq C_1 \Rightarrow \sup\{|\eta| : \eta \in \partial\varphi(x)\} \leq (R/r)C,$$

$$(3.2) \quad \varphi(x) \geq C_1 \Rightarrow \inf\{|\eta| : \eta \in \partial\varphi(x)\} \geq C.$$

PROOF. We first prove the following equivalence

$$\eta \in \partial\varphi(x) \quad \Leftrightarrow \quad (\eta = cw, \quad w \in \partial M_K(x), \quad c = f'(M_K(x))) .$$

We clearly have  $\partial\varphi(0) = \{0\}$ . For  $x \neq 0$ , take  $\gamma \in (0, 1)$ . Then, for  $\eta \in \partial\varphi(x)$  and for all  $y \in \mathcal{D}(\varphi)$ , we have

$$\langle \eta, x - (x - \gamma(x - y)) \rangle \geq \varphi(x) - \varphi(x - \gamma(x - y)) ,$$

hence

$$\langle \eta, x - y \rangle \geq \frac{1}{\gamma} (f(M_K(y)) - f(M_K(x)) - \gamma(M_K(x) - M_K(y))) .$$

Letting  $\gamma$  tend to 0, we obtain that  $\frac{\eta}{f'(M_K(x))} \in \partial M_K(x)$ . Conversely, for  $w \in \partial M_K(x)$ , we have

$$\langle f'(M_K(x))w, x - y \rangle \geq f'(M_K(x))(M_K(x) - M_K(y)) \geq f(M_K(x)) - f(M_K(y)),$$

which we wanted to prove.

Let now  $C$  be a given positive constant. For all  $w \in \partial M_K(x)$  and  $x \neq 0$  we have  $(1/R) \leq |w| \leq (1/r)$ . From this we deduce that if  $f'(M_K(x))|w| < C$ , then  $f'(M_K(x)) < CR$ , and so  $M_K(x) < (f')^{-1}(CR)$  and  $\varphi(x) < f((f')^{-1}(CR))$ . We can choose  $C_1 = f((f')^{-1}(CR))$ , and (3.2) is proved. Suppose now that  $f(M_K(x)) \leq C_1$ . Then we have  $M_K(x) \leq (f')^{-1}(CR)$ , hence  $f'(M_K(x)) \leq CR$  and  $f'(M_K(x))|w| \leq (R/r)C$ , and (3.1) is proved.  $\blacksquare$

We conclude the proof of Proposition 3.1(c) by proving the following Proposition 3.3.

**Proposition 3.3.** *Let  $\varphi$  be as in Proposition 3.1(c). Then Hypothesis 2.1(xi) is satisfied with  $D = R/r$ .*

PROOF. Consider some  $\zeta$  satisfying inclusion (2.10) with initial datum  $\zeta_0$ . Then, the following equality holds true for all  $t \in (0, \infty)$ :

$$|\alpha(t)\zeta_t|^2 + |g(t) - \alpha(t)\zeta_t|^2 + 2\alpha(t)\varphi(\zeta)_t = |g(t)|^2;$$

hence, we immediately deduce that, for all  $t \in (0, \infty)$ ,

$$(3.3) \quad \varphi(\zeta)_t = \frac{1}{2\alpha(t)} (|g(t)|^2 - |g(t) - \alpha(t)\zeta_t|^2 - |\alpha(t)\zeta_t|^2).$$

By assumption, we have  $|g(t)| \leq C$  and  $g(t) - \alpha(t)\zeta_t \in \partial\varphi(\zeta)$ . In view of (3.2), there is a constant  $C_1$  such that

$$\varphi(\zeta)_t \leq \frac{1}{2\alpha(t)} (C^2 - C^2 - |\alpha(t)\zeta_t|^2) \leq 0 \quad \forall t \in (0, \infty) \quad \text{if} \quad \varphi(\zeta) \geq C_1.$$

Hence,

$$(3.4) \quad \varphi(\zeta)_t(\varphi(\zeta) - C_1)^+ \leq 0 \quad \forall t \in (0, \infty).$$

Integrating from 0 to  $t$  and using the assumption  $\zeta_0 \in \mathcal{D}_C(\varphi)$ , it follows that  $\varphi(\zeta)(t) \leq C_1$  for all  $t \in (0, \infty)$ . Then, using (3.1), together with the fact that  $|g(t)| \leq C$  for all  $t \in (0, \infty)$ , we finally obtain that

$$(3.5) \quad |g(t) - \alpha(t)\zeta_t| \leq \frac{R}{r}C \quad \text{and} \quad |\alpha(t)\zeta_t| \leq \left(\frac{R}{r} + 1\right)C,$$

which concludes the proof.  $\blacksquare$

**Properties of the solution mapping under Hypothesis 2.1(xi).**

**Proposition 3.4.** *Let us consider the solutions  $\zeta_1, \zeta_2 \in W^{1,\infty}(0, \infty)$  to (2.10) associated with data  $\zeta_{01}, \zeta_{02} \in \mathcal{D}_C(\varphi)$ ,  $\alpha_1, \alpha_2 \in L^\infty_{\text{loc}}(0, \infty)$ , and with  $g_1, g_2 \in L^\infty(0, \infty)$  complying with the constraint*

$$|g_i(t)| \leq C, \quad i = 1, 2, \quad \text{a.e.},$$

*and let Hypothesis 2.1(xi) hold. Then there exists a constant  $L$  such that for every  $t \in (0, \infty)$  we have*

$$(3.6) \quad |\zeta_1 - \zeta_2|(t) \leq |\zeta_{01} - \zeta_{02}| + L \int_0^t \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |g_1 - g_2| \right) d\tau.$$

**PROOF.** Test the difference of the two inclusions (2.10) by  $\zeta_1 - \zeta_2$  and divide the resulting inequality by  $\alpha_1$ . Then we obtain for a.e.  $t \in (0, \infty)$ :

$$\langle \dot{\zeta}_1 - \dot{\zeta}_2, \zeta_1 - \zeta_2 \rangle \leq \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| |\langle \alpha_2 \dot{\zeta}_2, \zeta_1 - \zeta_2 \rangle| + |\langle g_1 - g_2, \zeta_1 - \zeta_2 \rangle|.$$

Using the bound for  $|\alpha_2 \dot{\zeta}_2|$  (cf. Hypothesis 2.1(xi), (2.11)), we get

$$\frac{d}{dt} |\zeta_1 - \zeta_2| \leq L \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |g_1 - g_2| \right)$$

from which (3.6) immediately follows by integrating over  $(0, t)$ . ■

**Proposition 3.5.** *Let Hypothesis 2.1(xi) hold, and let  $\zeta_n$  and  $\zeta$  be the solutions of (2.10) corresponding to the data  $(g_n, \alpha_n, \zeta_{0n})$  and  $(g, \alpha, \zeta_0)$ , respectively, with  $|g_n(t)| \leq C$ ,  $\zeta_{0n} \in \mathcal{D}_C(\varphi)$ . If  $\{\zeta_{0n}\}$  converges to  $\zeta_0$  in  $\mathbb{R}^d$ ,  $\{g_n\}$  converges to  $g$  and  $\{\alpha_n\}$  converges to  $\alpha$  in  $L^2(0, T)$  for some  $T > 0$ , then  $\{\zeta_n\}$  converges strongly to  $\zeta$  in  $L^2(0, T)$ .*

**PROOF.** Test (2.10), written for  $\zeta_n$ , by  $\dot{\zeta}_n$  in order to obtain

$$(g_n - \alpha_n \dot{\zeta}_n) \dot{\zeta}_n = \frac{d}{dt} \varphi(\zeta_n).$$

Now set  $\eta_n = \frac{g_n}{\sqrt{\alpha_n}} - 2\sqrt{\alpha_n} \dot{\zeta}_n$ . Then, by straightforward computations, we obtain that

$$\left| \frac{g_n}{\sqrt{\alpha_n}} \right|^2 - |\eta_n|^2 = 4 \frac{d}{dt} \varphi(\zeta_n).$$

We know, by Proposition 3.4, that  $\{\zeta_n\}$  converges uniformly to  $\zeta$  and that  $\varphi$  is Lipschitz continuous on  $\mathcal{D}_C(\varphi)$ . Hence, integrating over  $(0, t)$ , we obtain that  $|\eta_n|_{L^2(0, T)} \rightarrow |\eta|_{L^2(0, T)}$ . Since we know that  $\eta_n \rightarrow \eta$  weakly in  $L^2(0, T)$  (since  $\dot{\zeta}_n \rightarrow \dot{\zeta}$  weakly in  $L^2(0, T)$ ), we infer  $\eta_n \rightarrow \eta$  strongly in  $L^2(0, T)$ , which is sufficient in order to conclude the desired convergence. ■

### Examples of functions complying with Hypothesis 2.3(ii).

**Proposition 3.6.** *The function  $\varphi$  introduced in Hypothesis 2.1(i) satisfies Hypothesis 2.3(ii) in each of the following cases:*

- (a) if  $d = 1$ ;
- (b) if, for any  $C > 0$ ,  $\varphi$  is a  $C^1$ -function with Lipschitz continuous derivative on  $\mathcal{D}_C(\varphi)$ ;
- (c) if  $\varphi = I_K$ , where  $K$  is either a polyhedron or a smooth convex set with nonempty interior.

**PROOF.** The proofs of (a) and (c) follow respectively from [18, Prop. 3.4] and [7, Thm. 7.1, p. 88]. We briefly show here how to proceed to prove case (b). Let us consider solutions  $\zeta_1, \zeta_2 \in W^{1,\infty}(0, T)$  to (2.10) associated with the data  $\zeta_{01}, \zeta_{02} \in \mathcal{D}_C(\varphi)$ ,  $\alpha_1, \alpha_2 \in L^\infty(0, T)$ , and with  $g_1, g_2 \in L^\infty(0, T)$  complying with the constraint

$$|g_i(t)| \leq C, \quad i = 1, 2, \quad \text{a.e. in } (0, T).$$

By Hypothesis 2.1(xi),  $\zeta_1, \zeta_2$  remain in  $\mathcal{D}_{DC}(\varphi)$ . Using the Lipschitz continuity of  $\varphi'$  on  $\mathcal{D}_{DC}(\varphi)$ , we obtain that there exists a positive constant  $Q$  such that the following inequality holds true a.e.:

$$\alpha_1 |\dot{\zeta}_1 - \dot{\zeta}_2| \leq |\alpha_1 - \alpha_2| |\dot{\zeta}_2| + Q |\zeta_1 - \zeta_2| + |g_1 - g_2|.$$

Dividing by  $\alpha_1$ , and using the bound for  $|\alpha_2 \dot{\zeta}_2|$ , from Hypothesis 2.1(xi) (cf. (2.11)), we get

$$(3.7) \quad \begin{aligned} |\dot{\zeta}_1 - \dot{\zeta}_2| &\leq (C + DC) \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + Q |\zeta_1 - \zeta_2| + |g_1 - g_2| \\ &\leq M \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |\zeta_1 - \zeta_2| + |g_1 - g_2| \right) \end{aligned}$$

for some positive constant  $M$  (depending on  $C, D, Q$ ). We can rewrite this inequality in the following convenient form, for  $t \in (0, T)$ ,

$$\frac{d}{dt} (e^{-Mt} |\zeta_1 - \zeta_2|) \leq e^{-Mt} \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |g_1 - g_2| \right).$$

Integrating over  $(0, t)$ , and using the previous inequality (3.7), we deduce

$$|\dot{\zeta}_1 - \dot{\zeta}_2|(t) \leq L \left( |\zeta_{01} - \zeta_{02}| + \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |g_1 - g_2| + \int_0^t \left( \left| \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right| + |g_1 - g_2| \right) d\tau \right)$$

for some positive constant  $L$  (depending on  $C, D, Q$ ). Integrating once more in time we arrive at the desired inequality (2.17).  $\blacksquare$

**Proposition 3.7.** *Let  $f : [0, f_0) \rightarrow [0, +\infty)$  be an increasing, convex function with locally Lipschitz continuous derivative,  $f(0) = f'(0) = 0$ , and let  $K$  be a closed, convex set of class  $C^{1,1}$  such that  $B_r(0) \subset K \subset B_R(0)$ . Then,  $\varphi(x) = f(M_K(x))$  has a Lipschitz continuous derivative on  $\mathcal{D}_C(\varphi)$  for any  $C > 0$ , i.e., property (b) in Proposition 3.6 is satisfied.*

PROOF. Let  $C > 0$  be given. We denote  $\mathcal{D}_C f = \{s \in (0, f_0) : f'(s) \leq RC\}$ , and let  $L_C$  be the Lipschitz constant of  $f'$  on  $\mathcal{D}_C f$ . For  $x \in \mathcal{D}_C(\varphi)$  we have  $|\varphi'(x)| \leq C$ , hence  $f'(M_K(x)) \leq RC$ , that is,  $M_K(x) \in \mathcal{D}_C f$ . We now estimate the difference  $|\varphi'(x) - \varphi'(y)|$  on  $\mathcal{D}_C(\varphi)$ . Assume first that  $x \neq 0, y = 0$ . Then

$$\begin{aligned} |\varphi'(x) - \varphi'(y)| &= |\varphi'(x)| = f'(M_K(x))|M'_K(x)| \leq \frac{1}{r}f'(M_K(x)) \\ &\leq \frac{1}{r}L_C M_K(x) \leq \frac{1}{r^2}L_C|x| = \frac{L_C}{r^2}|x - y|. \end{aligned}$$

Consider now the case  $x \neq 0, y \neq 0$  and set  $J_K(x) = M_K(x)M'_K(x)$ ,  $J_K(y) = M_K(y)M'_K(y)$ . The mapping  $J_K$  is Lipschitz continuous on  $\mathbb{R}^d$  (with Lipschitz constant  $L_J$ ) (see [7, Section 5.2]), and we have

$$\begin{aligned} |\varphi'(x) - \varphi'(y)| &= |f'(M_K(x))M'_K(x) - f'(M_K(y))M'_K(y)| \leq \frac{f'(M_K(x))}{M_K(x)}|J_K(x) - J_K(y)| \\ &\quad + |M'_K(y)|\frac{f'(M_K(x))}{M_K(x)}|M_K(x) - M_K(y)| + |M'_K(y)||f'(M_K(x)) - f'(M_K(y))| \\ &\leq L_C \left( L_J + \frac{2}{r^2} \right) |x - y|, \end{aligned}$$

from which the assertion follows. ■

A relevant case for applications is, for example,  $\varphi(x) = -\log(1 - M_K^2(x))$ , see [11].

## 4 Existence of solutions

This section is devoted to the proof of the existence result stated in Section 2. We use a technique based on approximations, a priori estimates, and passage to the limit.

Let us first write down our equations (2.1) and (2.2) as

$$\begin{aligned} (4.1) \quad &\langle (e(\theta, \chi))_t, z \rangle + \int_{\Omega} k(\theta, \chi) \nabla \theta \cdot \nabla z \, dx + \int_{\partial\Omega} \gamma(\theta - \theta_T) z \, dA \\ &= - \int_{\Omega} (\lambda'(\chi)\chi_t + \beta(\varphi(\chi))_t + b[\chi]\chi_t) z \, dx \quad \forall z \in V, \quad \text{a.e. in } (0, \infty), \end{aligned}$$

$$(4.2) \quad \mu(\theta)\chi_t + (\beta + \theta)\partial\varphi(\chi) \ni -\lambda'(\chi) - \theta\sigma'(\chi) - b[\chi] - e_{\chi}(\theta, \chi) + \theta s_{\chi}(\theta, \chi) \quad \text{a.e. in } Q_{\infty}.$$

### 4.1 Approximation

Assuming Hypothesis 2.1 to hold, we proceed as follows: first we extend the domain of definition of  $c_V(\theta, \chi)$  by putting  $\tilde{c}_V(\theta, \chi) = c_V(|\theta|, \chi)$  for  $(\theta, \chi) \in \mathbb{R} \times \mathcal{D}(\varphi)$ , and set

$$\tilde{e}(\theta, \chi) = \int_0^{\theta} \tilde{c}_V(\xi, \chi) \, d\xi \quad \text{for } (\theta, \chi) \in \mathbb{R} \times \mathcal{D}(\varphi).$$

We now fix a truncation parameter  $\varrho \geq 1$ , which will be determined below, and define

$$\begin{aligned}\tilde{\mu}_\varrho(\theta) &= \begin{cases} \mu(|\theta|) & \text{if } |\theta| \leq \varrho \\ \mu(\varrho)(|\theta| - \varrho) & \text{if } |\theta| \geq \varrho \end{cases}, \\ s_\chi^\varrho(\theta, \chi) &= \begin{cases} \int_0^\theta \frac{(\tilde{c}_V)_\chi(\xi, \chi)}{\xi} d\xi & \text{if } |\theta| \leq \varrho \\ \int_0^\varrho \frac{(\tilde{c}_V)_\chi(\xi, \chi)}{\xi} d\xi & \text{if } |\theta| \geq \varrho \end{cases}.\end{aligned}$$

We fix an arbitrary  $T > 0$ , and split the interval  $[0, T]$  into an equidistant partition  $0 = t_0, t_1, \dots, t_n$ ,  $t_j = jT/n$  for  $j = 0, 1, \dots, n$ ,  $n \in \mathbb{N}$ , with the intention to let  $n$  tend to  $\infty$ . We choose sequences  $\{\theta_{0,n}\}_n \in V$  and  $\{\theta_{\Gamma,n}\}_n \in W^{1,2}(0, T; L^2(\partial\Omega))$  of approximate data such that  $\theta_{0,n}(x) \geq 1/n$  a.e.,  $\theta_{\Gamma,n}(x, t) \geq 1/n$  a.e.,  $\theta_{0,n} \rightarrow \theta_0$  strongly in  $H$ , and  $\theta_{\Gamma,n} \rightarrow \theta_\Gamma$  strongly in  $L^2(0, T; L^2(\partial\Omega))$ .

An approximate solution  $(\theta_n, \chi_n)$  will be constructed successively in intervals  $[t_{j-1}, t_j]$  for  $j = 1, \dots, n$ . Assuming that it is already known on  $[0, t_{j-1}]$ , we define

$$(4.3) \quad \bar{\chi}_n(x, t) = \chi_n(x, t_{j-1}) \quad x \in \Omega, \quad t \in (t_{j-1}, t_j), \quad j = 1, \dots, n,$$

$$\bar{\theta}_n(x, t) = \begin{cases} \theta_{0,n} & \text{for } t \in [0, t_1) \\ \frac{n}{T} \int_{t_{j-2}}^{t_{j-1}} \theta_n(x, \tau) d\tau & \text{for } t \in [t_{j-1}, t_j), \quad j \geq 2. \end{cases}$$

With this notation, we then state the following approximating problem. We use only the index  $n$  for the variables here (omitting the  $\varrho$  dependence), for simplicity.

**PROBLEM (P) $_{(n, \varrho)}$ .** Find two functions  $\theta_n \in H^1(0, T; H) \cap L^\infty(0, T; V)$  and  $\chi_n \in L^\infty(\Omega \times (0, T))^d$ ,  $\partial_t \chi_n \in L^\infty(\Omega \times (0, T))^d$ , such that  $\theta_n \geq \varepsilon_n$  a.e. in  $Q_T$  for some  $\varepsilon_n > 0$ ,  $\chi_n \in \mathcal{D}_C(\varphi)$ , and for all  $t \in (0, T)$  and  $z \in V$ , we have

$$(4.4) \quad \begin{aligned} & \int_\Omega \partial_t \left( \frac{1}{n} \theta_n(t) + \tilde{e}(\theta_n(t), \chi_n(t)) \right) z dx + \int_\Omega k(\bar{\theta}_n(t), \bar{\chi}_n(t)) \nabla \theta_n(t) \cdot \nabla z dx \\ & + \int_{\partial\Omega} \gamma(\theta_n(t) - \theta_{\Gamma,n}(t)) z dA = - \int_\Omega ((\lambda'(\chi_n)(t) + b[\chi_n](t)) \partial_t \chi_n(t) + \beta \partial_t(\varphi(\chi_n(t)))) z dx, \end{aligned}$$

$$(4.5) \quad \begin{aligned} & \tilde{\mu}_\varrho(\theta_n(t)) \partial_t \chi_n(t) + (\beta + |\theta_n(t)|) \partial \varphi(\chi_n)(t) \ni -\lambda'(\chi_n)(t) - |\theta_n(t)| \sigma'(\chi_n)(t) - b[\chi_n](t) \\ & - \tilde{e}_\chi(\theta_n(t), \chi_n(t)) + |\theta_n(t)| s_\chi^\varrho(\theta_n(t), \chi_n(t)) \quad \text{a.e. in } \Omega, \end{aligned}$$

with initial conditions

$$(4.6) \quad \theta_n(0) = \theta_{0,n}, \quad \chi_n(0) = \chi_0.$$

**Lemma 4.1.** *Under Hypothesis 2.1, for each  $\varrho > 0$  and  $n \in \mathbb{N}$  the PROBLEM (P) $_{(n, \varrho)}$  has a unique solution  $(\theta_n, \chi_n)$  with the required properties.*



**PROOF.** On each interval  $(t_{j-1}, t_j)$ , we can proceed as in the proof of [19, Thm. 2.2, p. 290]. We test a Galerkin approximation of (4.4) by the approximation of  $\partial_t \theta_n$ . The estimates are sufficient to pass to the limit in the Galerkin scheme and to obtain a solution on each interval  $(t_{j-1}, t_j)$ . We only have to check that the initial conditions at  $t_1, t_2, \dots$  are well defined. Indeed, since on each interval  $(t_{j-1}, t_j)$  we have  $\theta_n \in H^1(t_{j-1}, t_j; H) \cap L^\infty(t_{j-1}, t_j; V)$ , we also obtain that  $t \mapsto \theta_n(t, \cdot)$  is weakly continuous in  $(t_{j-1}, t_j)$  for every  $j$  with values in  $V$ . Moreover,  $\chi_n$  is strongly continuous with values in  $L^\infty(\Omega)^d$ , and there exists a positive constant  $C$  (independent of  $n$ ) such that  $\chi_n(t, \cdot) \in \mathcal{D}_C(\varphi)$  on  $(t_{j-1}, t_j)$  for every  $j = 1, \dots, n$ . Hence, we can define the initial conditions at  $t = t_j$  by  $\theta_n(t_j) = \theta_n(t_j -)$ . ■

## 4.2 A priori estimates

In this subsection, we perform suitable a priori estimates (independent of  $n$ ) for the solution. In the following, we will denote by  $C$  any positive constant that depends only on the data of the problem but may vary from line to line. In particular, it will not depend on the truncation parameter  $\varrho$  and discretization parameter  $n$ . If such a dependence takes place, we use the symbol  $C_\varrho$  for a constant that depends on  $\varrho$ , but not on  $n$ . Again, the same symbols will denote constants that may differ from line to line.

Let us, for simplicity, in this subsection occasionally omit the indices  $n$  and write simply  $\theta, \chi$  instead of  $\theta_n, \chi_n$  if no confusion arises. We denote (note that  $\theta_n > 0$  and so  $\tilde{e} = e$ )

$$(4.7) \quad u_n(t) := \frac{1}{n} \theta_n(t) + e(\theta_n(t), \chi_n(t)) \quad \text{for } t \in (0, T).$$

**Estimate for  $\chi_t$ .** Equation (4.5) is of the form (2.10) with

$$(4.8) \quad \alpha(t) = \tilde{\alpha}(\theta) := \frac{\tilde{\mu}_\varrho(\theta)}{\beta + \theta} \geq \frac{\mu_0(1 + \theta)}{\beta + \theta} \geq \mu_0 \min \left\{ 1, \frac{1}{\beta} \right\},$$

$$(4.9) \quad g(t) = \ell[\theta, \chi] := -\frac{1}{\beta + \theta} (\theta \sigma'(\chi) + \lambda'(\chi) + b[\chi] + e_\chi(\theta, \chi) - \theta s_\chi^\varrho(\theta, \chi)).$$

First, let us note that, using (2.4), (2.7), and (2.8), we get

$$(4.10) \quad \begin{aligned} s_\chi^\varrho(\theta, \chi) &\leq \int_0^{\min\{\theta, \varrho\}} \frac{|(c_V)_\chi(\xi, \chi)|}{\xi} d\xi \leq c_1 \int_0^1 \frac{c_V(\xi, \chi)}{\xi} d\xi + c_1 \int_1^\varrho \frac{\bar{c}}{\xi} d\xi \\ &\leq c_1 s(1, \chi) + c_1 \bar{c} \log \varrho \leq c_1^2 + c_1 \bar{c} \log \varrho. \end{aligned}$$

Hence, owing to Hypothesis 2.1, we have

$$\begin{aligned} |\ell[\theta, \chi]| &\leq C_\sigma + \frac{1}{\beta} (C_\lambda + C_b) + \frac{\theta}{\beta + \theta} \sup_{0 \leq \xi \leq \theta} |(c_V)_\chi(\xi, \chi)| + \frac{\theta}{\beta + \theta} |s_\chi^\varrho(\theta, \chi)| \\ &\leq C_\sigma + \frac{1}{\beta} (C_\lambda + C_b) + c_1 \sup_{0 \leq \xi \leq \theta} |c_V(\xi, \chi)| + c_1^2 + c_1 \bar{c} \log \varrho \\ &\leq C_\sigma + \frac{1}{\beta} (C_\lambda + C_b) + c_1 \bar{c} + c_1^2 + c_1 \bar{c} \log \varrho, \end{aligned}$$

where  $C_b$  denotes here the upper bound for the operator  $b$  defined in (1.13). Let us set

$$(4.11) \quad C_{\ell, \varrho} := C_\sigma + \frac{1}{\beta} (C_\lambda + C_b) + c_1 \bar{c} + c_1^2 + c_1 \bar{c} \log \varrho.$$

Then the conditions of Hypothesis 2.1(xi) are satisfied with the choice  $C = \max\{C_{\ell, \varrho}, C_0\}$ , where  $C_0$  is defined in Hypothesis 2.1(vii) and  $C_{\ell, \varrho}$  is defined in (4.11). Hence, we obtain the following estimates on  $\chi$ :

$$(4.12) \quad |\chi_n|_{L^\infty(Q_T)} + |\partial_t \chi_n|_{L^\infty(Q_T)} + |\partial_t(\varphi(\chi_n))|_{L^\infty(Q_T)} \leq C(1 + \log \varrho)^2,$$

where now  $C$  is a constant independent of  $\varrho$ .

**Estimate for  $\theta$ .** Taking  $z = \theta_n$  in (4.4), we get

$$(4.13) \quad \begin{aligned} & \int_{\Omega} \partial_t \left( \frac{1}{n} \theta(t) + e(\theta(t), \chi(t)) \right) \theta(t) \, dx + \int_{\Omega} k(\bar{\theta}_n(t), \bar{\chi}_n(t)) \nabla \theta(t) \cdot \nabla \theta(t) \, dx \\ & + \int_{\partial\Omega} \gamma(\theta(t) - \theta_{\Gamma, n}(t)) \theta(t) \, dA \\ & = - \int_{\Omega} (\lambda'(\chi)(t) \partial_t \chi(t) + \beta \partial_t(\varphi(\chi(t))) + b[\chi](t) \partial_t \chi(t)) \theta(t) \, dx. \end{aligned}$$

Define

$$U(\theta, \chi) = \int_0^\theta c_V(v, \chi) v \, dv \quad \text{for } (\theta, \chi) \in (0, \infty) \times \mathcal{D}(\varphi).$$

We have  $\partial_t U(\theta, \chi) = \theta \partial_t e(\theta, \chi) + (\partial_\chi U - \theta \partial_\chi e) \chi_t$ . We integrate (4.13) from 0 to  $t$  and rewrite the first term as follows:

$$\begin{aligned} & \int_0^t \int_{\Omega} \partial_t \left( \frac{1}{n} \theta(\tau) + e(\theta(\tau), \chi(\tau)) \right) \theta(\tau) \, d\tau \, dx \\ & = \int_{\Omega} \left( \frac{1}{2n} \theta^2(t) + U(\theta(t), \chi(t)) \right) \, dx - \int_{\Omega} \left( \frac{1}{2n} \theta^2(0) + U(\theta(0), \chi(0)) \right) \, dx \\ & \quad - \int_0^t \int_{\Omega} (\partial_\chi U(\theta(\tau), \chi(\tau)) - \theta(\tau) \partial_\chi e(\theta(\tau), \chi(\tau))) \chi_t(\tau) \, d\tau \, dx. \end{aligned}$$

There exist two constants  $C_1, C_2$  such that  $U(\theta, \chi) \geq C_1 \theta^2 - C_2$ . Hence, by (4.12) and the Gronwall's lemma, we obtain

$$(4.14) \quad \|\theta_n\|_{L^2(0, T; V) \cap L^\infty(0, T; H)} \leq C(1 + \log \varrho)^2.$$

By comparison, we also deduce that

$$(4.15) \quad \|\partial_t u_n\|_{L^2(0, T; V')} \leq C(1 + \log \varrho)^2.$$

**Estimate for  $\nabla\chi$ .** The function

$$\theta \mapsto \frac{\beta + \theta}{\tilde{\mu}_\varrho(\theta)}$$

is Lipschitz continuous in  $\mathbb{R}$  due to Hypothesis 2.1(v) and, with the help of the mean value theorem, it is straightforward to deduce that

$$\begin{aligned}
(4.16) \quad |\ell[\theta_1, \chi_1] - \ell[\theta_2, \chi_2]| &\leq |\sigma'(\chi_1) - \sigma'(\chi_2)| + \frac{C_\sigma}{\beta} |\theta_1 - \theta_2| + \frac{1}{\beta} |\lambda'(\chi_1) - \lambda'(\chi_2)| \\
&+ \frac{C_\lambda}{\beta^2} |\theta_1 - \theta_2| + \frac{1}{\beta} |b[\chi_1] - b[\chi_2]| + \frac{C_b}{\beta^2} |\theta_1 - \theta_2| + \frac{1}{\beta + \theta_1} \int_0^{\theta_1} c_1 |\chi_1 - \chi_2| \, d\xi \\
&+ \left| \frac{1}{\beta + \theta_1} \int_0^{\theta_1} (c_V)_\chi(\xi, \chi_2) \, d\xi - \frac{1}{\beta + \theta_2} \int_0^{\theta_2} (c_V)_\chi(\xi, \chi_2) \, d\xi \right| \\
&+ \frac{1}{\beta} |s_\chi^e(\theta_1, \chi_1) - s_\chi^e(\theta_2, \chi_2)| + \frac{|s_\chi^e(\theta_2, \chi_2)|}{\beta^2} |\theta_1 - \theta_2| \\
&\leq |\sigma'(\chi_1) - \sigma'(\chi_2)| + \frac{1}{\beta} (|\lambda'(\chi_1) - \lambda'(\chi_2)| + |b[\chi_1] - b[\chi_2]|) \\
&+ \frac{1}{\beta^2} (C_\sigma \beta + C_\lambda + C_b) |\theta_1 - \theta_2| + \frac{c_1}{\beta} |\chi_1 - \chi_2| + \frac{1}{\beta} (c_1 \bar{c} + c_1 \bar{c}) |\theta_1 - \theta_2| \\
&+ \frac{c_1}{\beta} (|\theta_1 - \theta_2| + |\chi_1 - \chi_2|) + \frac{1}{\beta^2} (c_1^2 + c_1 \log \varrho) |\theta_1 - \theta_2|.
\end{aligned}$$

Hence, we can apply estimate (3.6) in Proposition 3.4 to (4.5) with (for  $x, y \in \Omega$ )

$$\begin{aligned}
\alpha_1(t) &= \left( \frac{\tilde{\mu}_\varrho(\theta)}{\beta + \theta} \right) (x, t), \\
\alpha_2(t) &= \left( \frac{\tilde{\mu}_\varrho(\theta)}{\beta + \theta} \right) (y, t), \\
g_1(t) &= \ell[\theta(x, t), \chi(x, t)] = -\frac{1}{\beta + \theta} \left( \theta \sigma'(\chi) + \lambda'(\chi) + b[\chi] + e_\chi(\theta, \chi) \right. \\
&\quad \left. - \theta s_\chi^e(\theta, \chi) \right) (x, t), \\
g_2(t) &= \ell[\theta(y, t), \chi(y, t)] = -\frac{1}{\beta + \theta} \left( \theta \sigma'(\chi) + \lambda'(\chi) + b[\chi] + e_\chi(\theta, \chi) \right. \\
&\quad \left. - \theta s_\chi^e(\theta, \chi) \right) (y, t),
\end{aligned}$$

and

$$\zeta_1 = \chi(x, t), \quad \zeta_2 = \chi(y, t), \quad x, y \in \Omega.$$

Hence, we obtain

$$\begin{aligned}
(4.17) \quad |\chi(x, t) - \chi(y, t)| &\leq |\chi_0(x) - \chi_0(y)| + \hat{L}_\varrho \left( \int_0^t |\theta(x, s) - \theta(y, s)| \, ds \right. \\
&\quad \left. + \int_0^t (|b[\chi](x, s) - b[\chi](y, s)| + |\chi(x, s) - \chi(y, s)|) \, ds \right),
\end{aligned}$$

where  $\hat{L}_\varrho$  depends on  $L$ ,  $L_\mu$ , and the constants on the right-hand side of (4.16). Now, recalling (1.13), we have, by (4.16), that

$$\begin{aligned}
 (4.18) \quad |\chi(x, t) - \chi(y, t)| &\leq |\chi_0(x) - \chi_0(y)| + \hat{L}_\varrho \int_0^t |\theta(x, s) - \theta(y, s)| \, ds \\
 &\quad + 2\hat{L}_\varrho \int_0^t \int_\Omega |\kappa(x, z) (G'(\chi(x, s) - \chi(z, s)) - G'(\chi(y, s) - \chi(z, s)))| \, dz \, ds \\
 &\quad + 2\hat{L}_\varrho \int_0^t \int_\Omega |G'(\chi(y, s) - \chi(z, s)) (\kappa(x, z) - \kappa(y, z))| \, dz \, ds \\
 &\quad + \hat{L}_\varrho \int_0^t |\chi(x, s) - \chi(y, s)| \, ds.
 \end{aligned}$$

Thus, in view of Hypothesis 2.1(iii), we obtain that

$$\begin{aligned}
 (4.19) \quad |\chi(x, t) - \chi(y, t)| &\leq |\chi_0(x) - \chi_0(y)| + \hat{L}_\varrho \int_0^t |\theta(x, s) - \theta(y, s)| \, ds \\
 &\quad + \hat{L}_\varrho (2L_b + 1) \int_0^t |\chi(x, s) - \chi(y, s)| \, ds + 2\hat{L}_\varrho L_b \int_\Omega |\kappa(x, z) - \kappa(y, z)| \, dz,
 \end{aligned}$$

where  $L_b$  is a constant depending on the Lipschitz constants of  $G$  and  $G'$ ,  $\|\kappa\|_{L^\infty(\Omega \times \Omega)}$ ,  $|\Omega|$ , and  $T$ . From (4.19), using the assumptions  $\chi_0 \in \mathbf{V}$  and  $\kappa \in W^{1,\infty}(\Omega \times \Omega)$  (cf. Hypothesis 2.1(iii)), we immediately deduce that

$$|\nabla \chi(\cdot, t)| \leq C_\varrho \left( 1 + \int_0^t (|\nabla \theta(\cdot, s)| + |\nabla \chi(\cdot, s)|) \, ds \right) \quad \text{a.e. in } \Omega.$$

Now, with the help of Gronwall's lemma, we infer that

$$(4.20) \quad |\nabla \chi(\cdot, t)| \leq C_\varrho \left( 1 + \int_0^t |\nabla \theta(\cdot, s)| \, ds \right) \quad \text{a.e. in } \Omega.$$

Using finally (4.20) with (4.14), we get the desired estimate

$$(4.21) \quad \|\chi_n\|_{L^\infty(0,T;\mathbf{V})} \leq C_\varrho,$$

where  $C_\varrho$  denotes a positive constant depending increasingly on  $\varrho$ . From the definition (4.7) of  $u$ , it also follows that

$$(4.22) \quad \|u_n\|_{L^2(0,T;V)} \leq C_\varrho.$$

### 4.3 Lower and upper bounds on $\theta$

In this subsection, we first prove a bound for  $\log \theta$  entailing the strict positivity of the absolute temperature (in the limit when  $n \rightarrow \infty$ ). Then, we prove a (time dependent) upper bound holding true for the solution component  $\theta_n$  for  $n$  fixed, which enables us to proceed with the Moser iteration procedure in order to prove a uniform (independent of time, of  $n$ , and of  $\varrho$ ) upper bound on  $\theta$ . This permits us to remove the truncation parameter and to conclude the existence proof. Finally, we will prove a lower bound (independent of  $n$ ) on  $\theta$  holding true under the additional Hypothesis 2.3 that we will use for the proof of uniqueness of solutions.

**Estimate on  $\log \theta$ .** Let us rewrite equation (4.4), by using (4.5), in the following form, for all  $z \in V$ ,

$$(4.23) \quad \int_{\Omega} \partial_t \left( \frac{1}{n} \theta + e(\theta, \chi) \right) z \, dx + \int_{\Omega} k(\bar{\theta}_n, \bar{\chi}_n) \nabla \theta \cdot \nabla z \, dx + \int_{\partial\Omega} \gamma(\theta - \theta_{\Gamma,n}) z \, dA \\ = \int_{\Omega} \tilde{\mu}_{\varrho}(\theta) \chi_t^2 + \theta \chi_t R(\theta, \chi) z \, dx,$$

where

$$R(\theta, \chi) := \sigma'(\chi) - s_{\chi}^{\varrho}(\theta, \chi) + \xi, \quad \xi \in \partial\varphi(\chi), \quad |\xi(x, t)| \leq C \text{ a.e.},$$

$C$  being defined in Hypothesis 2.1(xi). We prove now an estimate on  $\log \theta$  in  $L^2(0, T; V)$  by taking in (4.4)  $z = T(\theta)$ , where

$$(4.24) \quad T(\theta) := - \left( 1 - \frac{1}{\theta} \right)^{-} = \begin{cases} 1 - \frac{1}{\theta} & \text{for } \theta \leq 1 \\ 0 & \text{for } \theta \geq 1 \end{cases}.$$

Notice that we are allowed to perform this estimate, with fixed  $n$ , because  $\theta \geq \varepsilon_n > 0$  a.e. for all  $n$  (cf. Lemma 4.1). We get, using equation (4.5) and Hypothesis 2.1(iv),

$$(4.25) \quad \frac{d}{dt} E(\theta, \chi) - \int_{\Omega} \chi_t \int_1^{\theta} (c_V)_{\chi}(\xi, \chi) T(\xi) \, d\xi \, dx + k_0 \|\nabla(\log \theta)^{-}\|_H^2 \\ + \int_{\partial\Omega} (\theta - \theta_{\Gamma,n}) T(\theta) \, dA \leq \int_{\Omega} \tilde{\mu}_{\varrho}(\theta) |\chi_t|^2 T(\theta) \, dx + \int_{\Omega} \theta T(\theta) \chi_t R(\theta, \chi) \, dx,$$

where

$$E(\theta, \chi) = - \int_{\Omega} \int_1^{\theta} c_V(\xi, \chi) \left( 1 - \frac{1}{\xi} \right)^{-} \, d\xi \geq 0.$$

Note that the first term on the right-hand side of (4.25) is nonpositive, while the other term can be estimated using estimates (4.10), (4.12), and (4.14) on our solution  $(\theta, \chi)$ . Regarding the second term on the left-hand side in (4.25), using (2.7) in Hypothesis 2.1(vi), we obtain that

$$\left| - \int_{\Omega} \chi_t \int_1^{\theta} (c_V)_{\chi}(\xi, \chi) T(\xi) \, d\xi \, dx \right| \leq c_1 E(\theta, \chi) \|\partial_t \chi\|_{L^{\infty}(\Omega)^d}.$$

Moreover, we treat the boundary integral in the following way:

$$(4.26) \quad \int_{\partial\Omega} (\theta - \theta_{\Gamma,n}) T(\theta) \, dA \geq \int_{\partial\Omega} \Psi(\theta) - \Psi(\theta_{\Gamma,n}) \, dA,$$

where  $\Psi$  is defined as

$$\Psi(\theta) := \begin{cases} \theta - \log \theta & \text{for } \theta \leq 1 \\ 1 & \text{for } \theta \geq 1 \end{cases},$$

and is a convex function on  $[0, +\infty)$  such that  $\Psi'(\theta) = T(\theta)$  for all  $\theta \in [0, +\infty)$ . Using estimates (4.12) and (4.14), and Hypothesis 2.1(x), we deduce that

$$(4.27) \quad E(\theta(x, t), \chi(x, t)) + k_0 \int_0^t \|\nabla(\log \theta)^{-}\|_H^2 \, d\xi + \int_0^t \int_{\partial\Omega} \Psi(\theta) \, dA \, d\xi \\ \leq C_{\varrho} \left( 1 + \int_0^t E(\theta, \chi) \right) + E(\theta_{0,n}, \chi_0) + \int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n}) \, dA \, d\xi$$

for a.e.  $(x, t) \in Q_T$ . Now, in view of (2.8), we have that

$$(4.28) \quad \begin{aligned} E(\theta_{0,n}, \chi_0) &= - \int_{\Omega} \int_1^{\theta_{0,n}} c_V(\xi, \chi_0) \left(1 - \frac{1}{\xi}\right)^- d\xi dx \\ &\leq \int_{\Omega} \int_0^1 c_V(\xi, \chi_0) \left(\frac{1}{\xi} - 1\right) d\xi dx \leq \int_{\Omega} s(1, \chi_0) \leq c_1 |\Omega|. \end{aligned}$$

Moreover, we can estimate the term  $\int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n}) dA d\tau$  using Hypothesis 2.1(x) as follows:

$$(4.29) \quad \int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n}) dA d\xi \leq C (\|\theta_{\Gamma,n}\|_{L^\infty(\Sigma_t)} + \|(\log(\theta_{\Gamma,n}))^-\|_{L^1(\Sigma_t)}) \leq C.$$

Using a standard Gronwall's lemma in (4.27), together with the estimates (4.29) and (4.28), we obtain the desired bound

$$\|(\log \theta_n)^-\|_{L^2(0,T;V)} \leq C_\varrho,$$

which, together with estimate (4.14) for  $\theta$  in  $L^2(0, T; V)$ , gives

$$(4.30) \quad \|\log \theta_n\|_{L^2(0,T;V)} \leq C_\varrho.$$

**Lower bound for  $\theta$  under Hypothesis 2.3 (iii)–(vi).** Let us consider relation (4.23). Notice that, by the previous estimates, it follows that there exists a positive constant  $R_\varrho$  such that  $|R(\theta, \chi)| \leq R_\varrho$  a.e. in  $Q_T$ . Hence, for every  $z \in V$  such that  $z \geq 0$  a.e., we obtain the following inequality (notice that by virtue of Lemma 4.1, we have here  $\theta_t \in L^2(0, T; H)$ ):

$$(4.31) \quad \begin{aligned} &\int_{\Omega} \left( \frac{1}{n} \theta_t + c_V(\theta, \chi) \theta_t \right) z dx + \int_{\Omega} k(\bar{\theta}_n, \bar{\chi}_n) \nabla \theta \cdot \nabla z dx \\ &= \int_{\Omega} \tilde{\mu}_\varrho(\theta) \left( \chi_t + \frac{R(\theta, \chi) \theta}{2 \tilde{\mu}_\varrho(\theta)} \right)^2 - \frac{R^2(\theta, \chi) \theta^2}{4 \tilde{\mu}_\varrho(\theta)} z dx \geq - \int_{\Omega} \frac{R_\varrho^2 \theta^2}{4 \tilde{\mu}_\varrho(\theta)} z dx. \end{aligned}$$

We now compare this inequality with the following ODE:

$$(4.32) \quad \tilde{c}(w) w_t = - \frac{R_\varrho^2 w^2}{4 \tilde{\mu}_\varrho(w)}, \quad w(0) = w_0,$$

where  $\tilde{c}$  is defined in Hypothesis (2.3)(iii) and  $w_0 = \min_{x \in \Omega} \theta_0(x) \geq \theta_*$  (cf. Hypothesis 2.3(v)).

Notice that the solution  $w$  is decreasing and does not vanish in finite time, due to Hypothesis 2.3(iii). The function  $w$  does not depend on  $x$ , hence we may add to the ODE in (4.32) the term  $-\operatorname{div}(k(\bar{\theta}_n, \bar{\chi}_n) \nabla w)$ , which is equal to 0. Using the fact that  $w_t < 0$  and  $\tilde{c}(w) \leq \frac{1}{n} + c_V(w, \chi)$ , we obtain, subtracting (4.31) from (4.32), the inequality

$$\begin{aligned} &\int_{\Omega} \left( \left( \frac{1}{n} + c_V(w, \chi) \right) w_t - \left( \frac{1}{n} + c_V(\theta, \chi) \right) \theta_t \right) z dx \\ &+ \int_{\Omega} k(\bar{\theta}_n, \bar{\chi}_n) \nabla(w - \theta) \cdot \nabla z dx \leq \frac{R_\varrho^2}{4} \int_{\Omega} \left( \frac{\theta^2}{\tilde{\mu}_\varrho(\theta)} - \frac{w^2}{\tilde{\mu}_\varrho(w)} \right) z dx. \end{aligned}$$

We now take as test function  $z = H_\varepsilon(w - \theta)$ , where  $H_\varepsilon$  is the regularization of the Heaviside function  $H$ ,

$$(4.33) \quad H_\varepsilon(v) = \begin{cases} 0 & \text{if } v \leq 0 \\ v/\varepsilon & \text{if } v \in (0, \varepsilon) \\ 1 & \text{if } v \geq \varepsilon \end{cases}.$$

By virtue of Hypothesis 2.3 (iv)–(v), we deduce

$$\int_{\Omega} \left( \left( \frac{1}{n} + c_V(w, \chi) \right) w_t - \left( \frac{1}{n} + c_V(\theta, \chi) \right) \theta_t \right) H_\varepsilon(w - \theta) \, dx \leq 0,$$

and we can pass to the limit in this inequality for  $\varepsilon \searrow 0$ , getting

$$\int_{\Omega} \left( \left( \frac{1}{n} + c_V(w, \chi) \right) w_t - \left( \frac{1}{n} + c_V(\theta, \chi) \right) \theta_t \right) H(w - \theta) \, dx \leq 0,$$

that is,

$$(4.34) \quad \frac{\partial}{\partial t} \int_{\Omega} \left( \left( \frac{1}{n} w + e(w, \chi) \right) - \left( \frac{1}{n} \theta + e(\theta, \chi) \right) \right)^+ \, dx \leq \int_{\Omega} (e_\chi(w, \chi) - e_\chi(\theta, \chi)) \chi_t H(w - \theta) \, dx.$$

Notice now that, by Hypothesis 2.1(vi) (cf. (2.4) and (2.7)), we have

$$|e_\chi(w, \chi) - e_\chi(\theta, \chi)| \leq \max_{\tau \leq w_0} (c_V(\tau, \chi)) |w - \theta| \leq \bar{c} |w - \theta|.$$

Integrating (4.34) over  $(0, t)$ , and using the boundedness of  $\chi_t$  in  $L^\infty(Q_T)$ , we obtain, by the choice of the initial data  $\theta_0$  and  $w_0$ , that

$$(4.35) \quad \int_{\Omega} \left( \left( \frac{1}{n} w + e(w, \chi) \right) - \left( \frac{1}{n} \theta + e(\theta, \chi) \right) \right)^+ (t) \, dx \leq \bar{c} \int_{\Omega} (w - \theta)^+ \, dx.$$

For  $w > \theta$ , we have

$$\frac{e(w, \chi) - e(\theta, \chi)}{w - \theta} = \frac{\int_{\theta}^w c_V(\tau, \chi) \, d\tau}{w - \theta} \geq \frac{\int_{\theta}^w \tilde{c}(\tau) \, d\tau}{w - \theta} \geq \frac{1}{w} \int_0^w \tilde{c}(\tau) \, d\tau =: \tilde{C}(w).$$

The function  $\tilde{C}$  is nondecreasing,  $\tilde{C}(w) > 0$  for  $w > 0$ . Hence,

$$(4.36) \quad \left( \frac{1}{n} w + e(w, \chi) \right) - \left( \frac{1}{n} \theta + e(\theta, \chi) \right) \geq \tilde{C}(w)(w - \theta) \quad \text{for } w \geq \theta.$$

Inequality (4.35) then yields

$$\tilde{C}(w(t)) \int_{\Omega} (w - \theta)^+(t) \, dx \leq C \int_0^t \int_{\Omega} (w - \theta)^+ \, dx \, ds$$

for every  $t \in (0, T)$ . From Gronwall's lemma we conclude that

$$(4.37) \quad \theta_n(x, t) \geq w(t) \quad \text{a.e. in } Q_T.$$

**Upper bound for  $\theta$ .** Let us denote the right-hand side in (4.4) by

$$M(\theta, \chi) := ((\lambda'(\chi)(t) + b[\chi](t))\partial_t \chi(t) + \beta \partial_t (\varphi(\chi(t))))$$

which, due to the previous estimates is bounded by a positive constant, say,  $\tilde{M}_\varrho$ . Then, we compare the inequality, for all  $z \in V$ ,  $z \geq 0$  a.e.,

$$(4.38) \quad \begin{aligned} & \int_{\Omega} \partial_t \left( \frac{1}{n} \theta(t) + e(\theta(t), \chi(t)) \right) z \, dx + \int_{\Omega} k(\bar{\theta}_n(t), \bar{\chi}_n(t)) \nabla \theta(t) \cdot \nabla z \, dx \\ & + \int_{\partial\Omega} \gamma(\theta(t) - \theta_{\Gamma,n}(t)) z \, dA \leq \int_{\Omega} \tilde{M}_\varrho z \, dx, \end{aligned}$$

with the following ODE:

$$(4.39) \quad \frac{1}{n} \dot{v}_n = \tilde{M}_\varrho, \quad v(0) = v_0,$$

where  $v_0 = \max\{\sup \theta_0, \sup \theta_\Gamma\}$  (cf. Hypothesis 2.1 (viii) (x)).

Then we have that

$$v_n(t) = v_0 + \tilde{M}_\varrho n t.$$

We proceed as above, adding to the ODE the term  $-\operatorname{div}(k(\bar{\theta}_n, \bar{\chi}_n) \nabla v_n)$ , which is equal to 0, and subtracting (4.39) from (4.38). We test the resulting inequality by  $H_\varepsilon(\theta - v_n)$  (cf. (4.33)). Using the fact that  $(1/n + c_V(w, \chi)) \dot{v}_n \geq 0$  and that  $\theta \in H^1(0, T; H)$ , we can let  $\varepsilon$  tend to 0, getting

$$\begin{aligned} & \int_{\Omega} \frac{\partial}{\partial t} \left( \left( \frac{1}{n} \theta + e(\theta, \chi) \right) - \left( \frac{1}{n} v + e(v, \chi) \right) \right)^+ \, dx \\ & - \int_{\Omega} (e_\chi(\theta, \chi) - e_\chi(v, \chi)) \chi_t H(\theta - v) \, dx \leq 0. \end{aligned}$$

Using now the Lipschitz continuity of  $e_\chi$  (cf. Hypothesis 2.1(vi)) and the boundedness of  $\chi_t$ , we obtain that

$$\int_{\Omega} \frac{\partial}{\partial t} \left( \left( \frac{1}{n} \theta + e(\theta, \chi) \right) - \left( \frac{1}{n} v + e(v, \chi) \right) \right)^+ \, dx \leq C_\varrho \int_{\Omega} (\theta - v)^+ \, dx.$$

Integrating over  $(0, t)$  and using the choice of the initial condition  $v_0$ , we infer

$$\int_{\Omega} (\theta - v)^+(t) \, dx \leq C_\varrho \int_0^t \int_{\Omega} (\theta - v)^+ \, dx \, d\tau,$$

and, applying Gronwall's lemma, we get the desired upper bound

$$(4.40) \quad \theta_n(x, t) \leq v_n(t) \quad \text{a.e. in } Q_T.$$



**Moser estimate.** In order to conclude the proof of existence of solutions to (2.1)–(2.3), it remains only to prove that the  $\theta$  component of the solution  $(\theta, \chi)$  is bounded from above independently of  $T$ ,  $\varrho$  and  $n$ .

This will enable us to choose  $\varrho$  sufficiently large in such a way that in this range of values of  $\theta$  we have  $s_\chi = s_\chi^\varrho$ . To this end, we perform the following Moser estimate.

We will make repeated use of the well-known interpolation inequality (cf. [3])

$$(4.41) \quad \|v\|_H \leq A \left( \eta \|\nabla v\|_H + \eta^{-N/2} \|v\|_{L^1(\Omega)} \right),$$

which holds for every  $v \in V$  and every  $\eta \in (0, 1)$ , with a positive constant  $A$  independent of  $v$  and  $\eta$ .

Following the ideas already exploited in [19, Prop. 3.10, p. 296], for  $j \in \mathbb{N}$ , we choose in (4.4)  $z = ((\theta_n - \theta_R)^+)^{2^j-1} \in L^2(0, T; H^1(\Omega))$ , with  $\theta_R = \max\{\Theta_\Gamma - 1, \Theta, 1\}$ , where

$$|\theta_{\Gamma,n}(x, t)| \leq \Theta_\Gamma \quad \text{a.e. in } \Sigma_T, \quad |\theta_{0,n}(x)| \leq \Theta \quad \text{a.e. in } \Omega.$$

We know that  $z \in L^2(0, T; H^1(\Omega))$ , due to the upper bound on  $\theta_n$  proved in (4.40).

Here below, we denote by  $C_i$ ,  $i = 1, 2, \dots$  some positive constants that may depend on the data of the problem, but not on  $j$ ,  $T$ ,  $\varrho$ , and  $n$ . We omit again the indices  $n$  in  $\theta_n$ ,  $\chi_n$ , for simplicity.

Now set  $u = (\theta - \theta_R)$  and take  $z = (u^+)^{2^j-1}$  in (4.4) to obtain that

$$(4.42) \quad \begin{aligned} & \left\langle \frac{1}{n} \theta_t(t) + (e(\theta(t), \chi(t)))_t, (u^+)^{2^j-1} \right\rangle + \int_\Omega k(\bar{\theta}_n(t), \bar{\chi}_n(t)) \nabla \theta(t) \cdot \nabla (u^+)^{2^j-1} dx \\ & + \int_{\partial\Omega} \gamma(\theta(t) - \theta_{\Gamma,n}) (u^+)^{2^j-1} dA \\ & = - \int_\Omega (\chi'(\chi)(t) \chi_t(t) + \beta(\varphi(\chi(t)))_t - b[\chi](t) \chi_t(t)) (u^+)^{2^j-1} dx. \end{aligned}$$

Our aim is to prove that there exists a positive constant  $C^*$  (independent of  $T$ ,  $\varrho$ , and  $n$ ) such that

$$(4.43) \quad \|\theta(t)\|_{L^\infty(\Omega)} \leq C^* (1 + \log \varrho)^{4+2N} \quad \text{for a.e. } t \in (0, T).$$

The first term on the left-hand side can be rewritten as

$$(4.44) \quad \begin{aligned} & \left\langle \frac{1}{n} \theta_t(t) + (e(\theta(t), \chi(t)))_t, (u^+)^{2^j-1} \right\rangle = \frac{1}{n} \langle \theta_t, (u^+)^{2^j-1} \rangle + \langle \theta_t, c_V(\theta, \chi)(u^+)^{2^j-1} \rangle \\ & + \langle e_\chi(\theta, \chi) \chi_t, (u^+)^{2^j-1} \rangle. \end{aligned}$$

Let us deal with the second term on the right-hand side in (4.44), using Hypothesis 2.1(vi) (notice that by (2.5), we have  $c_V \geq \underline{c}$  in the set where  $\theta \geq \theta_R \geq 1$ ) as follows:

$$(4.45) \quad \langle \theta_t, c_V(\theta, \chi)(u^+)^{2^j-1} \rangle = \frac{d}{dt} E_j(u, \chi) - \int_\Omega \left( \int_0^u (c_V)_\chi(\xi + \theta_R, \chi) (\xi^+)^{2^j-1} d\xi \right) \chi_t dx,$$

where

$$(4.46) \quad \underline{c} 2^{-j} \int_{\Omega} (u^+)^{2^j} dx \leq E_j(u, \chi) := \int_{\Omega} \int_0^u c_V(\xi + \theta_R, \chi) (\xi^+)^{2^j-1} d\xi dx \leq \bar{c} 2^{-j} \int_{\Omega} (u^+)^{2^j} dx.$$

Then, using (4.12) and Hypothesis 2.1(vi), we infer that

$$(4.47) \quad \int_{\Omega} \left( \int_0^u (c_V)_\chi(\xi + \theta_R, \chi) (\xi^+)^{2^j-1} d\xi \right) \chi_t dx \leq C_1(1 + \log \varrho)^2 E_j(u, \chi).$$

Moreover, by Hypothesis 2.1, we obtain the inequality

$$\begin{aligned} & \int_{\Omega} k(\bar{\theta}_n, \bar{\chi}_n) \nabla u \nabla \left( (u^+)^{2^j-1} \right) dx + \int_{\partial\Omega} \gamma (u - \theta_{\Gamma,n} + \theta_R) (u^+)^{2^j-1} dA \\ & \geq k_0 \frac{2^j - 1}{2^{2j-2}} \int_{\Omega} \left| \nabla \left( (u^+)^{2^j-1} \right) \right|^2 dx + \int_{\partial\Omega} \gamma \left( (u^+)^{2^j} - (u^+)^{2^j-1} \right) dA. \end{aligned}$$

Now, set  $\Phi_j = (u^+)^{2^j-1}$ . Regarding the terms on the right-hand side in (4.42) and the last term in (4.44), using (4.12) and Hypothesis 2.1(vi), we realize that

$$\begin{aligned} & - \int_{\Omega} (\lambda'(\chi)(t) \chi_t(t) + \beta(\varphi(\chi(t)))_t - b[\chi](t) \chi_t(t) - e_\chi(\theta, \chi) \chi_t) (u^+)^{2^j-1} dx \\ & \leq C_2(1 + \log \varrho)^2 \left( \int_{\Omega} |\Phi_j|^2 dx + 1 \right). \end{aligned}$$

Let us now set  $E_j^n = E_j + \frac{2^{-j}}{n} \int_{\Omega} |\Phi_j|^2 dx$ . Then, with the help of Hölder's and Young's inequalities, we deduce that

$$\begin{aligned} & \frac{d}{dt} E_j^n(u, \chi) + \frac{k_0(2^j - 1)}{2^{2j-2}} \int_{\Omega} |\nabla \Phi_j|^2 dx + \int_{\partial\Omega} \gamma |\Phi_j|^2 dA \\ & \leq (1 - 2^{-j}) \int_{\partial\Omega} \gamma |\Phi_j|^2 dA + 2^{-j} \int_{\partial\Omega} \gamma dA + C_1(1 + \log \varrho)^2 E_j(u, \chi) \\ & \quad + C_2(1 + \log \varrho)^2 \int_{\Omega} (|\Phi_j|^2 + 1) dx. \end{aligned}$$

Multiplying the above inequality by  $2^j$ , in view of the upper bound for  $E_j(u, \chi)$  in (4.46), we find out that

$$(4.48) \quad \begin{aligned} & 2^j \frac{d}{dt} E_j^n(u, \chi) + 2k_0 \int_{\Omega} |\nabla \Phi_j|^2 dx + \int_{\partial\Omega} \gamma |\Phi_j|^2 dA \\ & \leq 2^j C_3(1 + \log \varrho)^2 \left( 1 + \int_{\Omega} |\Phi_j|^2 dx \right). \end{aligned}$$

We now use the interpolation inequality (4.41) and note that, thanks to estimate (4.14), we have

$$\|\Phi_1\|_{L^1(\Omega)}^2 = \left( \int_{\Omega} u^+ dx \right)^2 \leq C_4(1 + \log \varrho)^4, \quad \|\Phi_j\|_{L^1(\Omega)}^2 = \|\Phi_{j-1}\|_H^4.$$

Thus, we derive the inequalities

$$(4.49) \quad \int_{\Omega} |\Phi_1|^2 dx \leq 2A^2 (\eta^2 \|\nabla \Phi_1\|_H^2 + \eta^{-N} C_4 (1 + \log \varrho)^4),$$

$$(4.50) \quad \int_{\Omega} |\Phi_j|^2 dx \leq 2A^2 (\eta^2 \|\nabla \Phi_j\|_H^2 + \eta^{-N} \|\Phi_{j-1}\|_H^4) \quad \text{for } j > 1.$$

For  $j = 1$ , we infer from (4.48) and (4.49) that

$$\begin{aligned} & 2 \frac{d}{dt} E_1^n(u, \chi) + 2k_0 \int_{\Omega} |\nabla \Phi_1|^2 dx + \int_{\partial\Omega} \gamma |\Phi_1|^2 dA \\ & \leq C_5 (1 + \log \varrho)^2 \left( 1 + \eta^2 \int_{\Omega} |\nabla \Phi_1|^2 dx + \eta^{-N} (1 + \log \varrho)^4 \right). \end{aligned}$$

Choosing  $\eta = \sqrt{k_0}/(\sqrt{C_5(1 + \log \varrho)^2})$ , we find that

$$(4.51) \quad 2 \frac{d}{dt} E_1^n(u, \chi) + k_0 \int_{\Omega} |\nabla \Phi_1|^2 dx + \int_{\partial\Omega} \gamma |\Phi_1|^2 dA \leq C_6 (1 + \log \varrho)^{6+N}.$$

For  $j > 1$ , we get

$$\begin{aligned} & 2^j \frac{d}{dt} E_j^n(u, \chi) + 2k_0 \int_{\Omega} |\nabla \Phi_j|^2 dx + \int_{\partial\Omega} \gamma |\Phi_j|^2 dA \\ & \leq 2^j C_7 (1 + \log \varrho)^2 \left( 1 + \eta^2 \int_{\Omega} |\nabla \Phi_j|^2 dx + \eta^{-N} \|\Phi_{j-1}\|_H^4 \right). \end{aligned}$$

Choosing  $\eta = \sqrt{k_0}/(\sqrt{2^j(C_7(1 + \log \varrho)^2)})$ , we conclude from (4.48) and (4.50) that

$$(4.52) \quad \begin{aligned} & 2^j \frac{d}{dt} E_j^n(u, \chi) + k_0 \|\nabla \Phi_j\|_H^2 + \int_{\partial\Omega} \gamma |\Phi_j|^2 dA \\ & \leq 2^j \left(\frac{N}{2} + 1\right) C_8 (1 + \log \varrho)^2 \left( 1 + (1 + \log \varrho)^N \|\Phi_{j-1}\|_H^4 \right). \end{aligned}$$

By assumption, we have  $\|\Phi_j(0)\|_H^2 = 0$ . Hence, integrating (4.51) and (4.52) with respect to time and using the lower bound in (4.46), we obtain that

$$\begin{aligned} & \|\Phi_1(t)\|_H^2 \leq C_9 (1 + \log \varrho)^{6+N}, \\ & \|\Phi_j(t)\|_H^2 \leq C_{10} 2^j \left(\frac{N}{2} + 1\right) (1 + \log \varrho)^2 \left( 1 + (1 + \log \varrho)^N \max_{0 \leq \tau \leq t} \|\Phi_{j-1}(\tau)\|_H^4 \right). \end{aligned}$$

Define now

$$z_j(t) = \max_{0 \leq \tau \leq t} \sqrt{\|u(\tau)\|_{L^{2^j}(\Omega)}} = \max_{0 \leq \tau \leq t} \|\Phi_j(\tau)\|_H^{2^{-j}}.$$

Then we have

$$\begin{aligned} & z_1(t) \leq C_{11} (1 + \log \varrho)^{(6+N)/4}, \\ & z_j(t) \leq C_{10}^{2^{-j-1}} 2^{j(\frac{N}{2} + 1)2^{-j-1}} (1 + \log \varrho)^{(N+2)2^{-j-1}} \max\{1, z_{j-1}(t)\}. \end{aligned}$$

In particular, putting  $y_j(t) = \max\{1, z_j(t)\}$ , we get

$$(4.53) \quad y_1(t) \leq C_{11}(1 + \log \varrho)^{(6+N)/4},$$

$$(4.54) \quad y_j(t) \leq \left(C_{12}(1 + \log \varrho)^{(\frac{N}{2}+1)}\right)^{2^{-j}} 2^{\frac{j}{2}(\frac{N}{2}+1)2^{-j}} y_{j-1}(t), \quad \text{for } j \geq 2.$$

Hence, passing to the logarithm in the inequality (4.54) and summing up the result from 2 to  $j$ , we obtain

$$\begin{aligned} \log y_j(t) &\leq \sum_{i=2}^j 2^{-i} \left( \log \left( C_{12}(1 + \log \varrho)^{\frac{N}{2}+1} \right) + \frac{i}{2} \left( \frac{N}{2} + 1 \right) \log 2 \right) + \log(y_1(t)) \\ &= \log \left( C_{12}(1 + \log \varrho)^{\frac{N}{2}+1} \right) \sum_{i=2}^j 2^{-i} + \left( \frac{N}{2} + 1 \right) \log 2 \sum_{i=2}^j i 2^{-i} \\ &\quad + \log \left( C_{11} (1 + \log \varrho)^{\frac{(6+N)}{4}} \right) \\ &\leq \log \left( 2^{(\frac{N}{2}+1)C_{13}} \right) + \log \left( C_{14} (1 + \log \varrho)^{N+2} \right), \end{aligned}$$

independently of  $j$  and  $t > 0$ . Hence, we deduce

$$y_j(t) \leq 2^{(\frac{N}{2}+1)C_{13}} \left( C_{14} (1 + \log \varrho)^{N+2} \right),$$

independently of  $j$  and  $t > 0$ . Choosing a proper  $\tilde{C}$ , which is independent of  $\varrho$ , we can conclude that

$$\sup_{t \geq 0, j \in \mathbb{N}} \sqrt{\|u(t)\|_{L^{2^j}(\Omega)}} \leq \tilde{C}(1 + \log \varrho)^{2+N}.$$

Formula (4.43) now immediately follows.

#### 4.4 Passage to the limit as $n \rightarrow \infty$

Our aim now is to pass to the limit in (4.4)–(4.6) as  $n \rightarrow \infty$ .

From (4.12), (4.14), (4.15), (4.21), (4.22), it follows that, up to the extraction of some subsequence of  $n$  as  $n \rightarrow \infty$ , there exist three functions  $u, \theta : (0, T) \rightarrow H$ ,  $\chi : (0, T) \rightarrow \mathbf{H}$ , such that we have (as a consequence of the generalized Ascoli theorem, see, e.g., [23, Cor. 8, p. 90])

$$(4.55) \quad \begin{aligned} u_n &\rightarrow u && \text{weak star in } H^1(0, T; V') \cap L^\infty(0, T; H) \cap L^2(0, T; V) \\ &&& \text{and strongly in } C^0([0, T]; V') \cap L^2(0, T; H), \end{aligned}$$

$$(4.56) \quad \theta_n \rightarrow \theta \quad \text{weak star in } L^\infty(0, T; H) \cap L^2(0, T; V),$$

$$(4.57) \quad \chi_n \rightarrow \chi \quad \text{weak star in } L^\infty(0, T; \mathbf{V}),$$

$$(4.58) \quad \partial_t \chi_n \rightarrow \partial_t \chi \quad \text{weak star in } L^\infty(\Omega \times (0, T))^d,$$

$$(4.59) \quad \chi_n \rightarrow \chi \quad \text{strongly in } C^0([0, T]; \mathbf{H}).$$

Moreover, as  $\chi_n$  are uniformly bounded, it is easy to see that

$$(4.60) \quad \bar{\chi}_n \rightarrow \chi \quad \text{weak star in } L^\infty(0, T; \mathbf{V}) \quad \text{and strongly in } L^q(Q_T),$$

for every  $q \in (1, \infty)$ , as  $n \rightarrow \infty$ . Note that (cf. (4.7))  $u \geq 0$  and  $\theta \geq 0$  a.e. Then it turns out that, at least for a subsequence,  $u_n \rightarrow u$  a.e. Now, we denote by  $\psi(\cdot, \chi)$  the inverse function of  $e$  with respect to the first variable, that is,  $e(\psi(w, \chi)) = w$  for all  $(w, \chi) \in [0, \infty) \times \mathcal{D}(\varphi)$ . Since  $e$  is continuous, increasing in  $\theta$ , and such that  $e(\theta, \chi) \geq e_1\theta - e_2$  for some constants  $e_1, e_2 > 0$ , we infer that  $\psi$  is continuous with a linear growth in  $[0, \infty) \times \mathcal{D}(\varphi)$  and  $\psi(0, \chi) = 0$ . The Nemytskii operator is therefore continuous in  $L^2(Q_T)$ , and so this function is continuous and increasing with a linear growth. Hence, we have that

$$\theta_n = \psi(u_n - \theta_n/n, \chi_n) \rightarrow \psi(u, \chi) \quad \text{strongly in } L^2(0, T; H).$$

Hence,  $\theta = \psi(u, \chi)$ , or, equivalently,  $u = e(\theta, \chi)$ . Finally, we check that also  $\bar{\theta}_n$  converge strongly to  $\theta$  in  $L^2(0, T; H)$ , at least for a subsequence of  $n \rightarrow \infty$ . Indeed, from the definition of  $\bar{\theta}_n$ , we get

$$(4.61) \quad \begin{aligned} \int_0^T \int_\Omega |\bar{\theta}_n(x, t) - \theta_n(x, t)|^2 dx dt &= \int_\Omega \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\bar{\theta}_n(x, t) - \theta_n(x, t)|^2 dt dx \\ &= \int_0^{T/n} \|\theta_n(t) - \theta_{0,n}\|_H^2 dt + \left(\frac{n}{T}\right)^2 \sum_{j=2}^n \int_\Omega \int_{t_{j-1}}^{t_j} \left| \int_{t_{j-2}}^{t_{j-1}} (\theta_n(x, t) - \theta_n(x, \tau)) d\tau \right|^2 dt dx. \end{aligned}$$

By (4.56), the first integral on the right-hand side of (4.61) can be estimated as follows:

$$\int_0^{T/n} \|\theta_n(t) - \theta_{0,n}\|_H^2 dt \leq \frac{C}{n}.$$

Regarding the second term in (4.61), we proceed as follows:

$$\begin{aligned} &\left(\frac{n}{T}\right)^2 \sum_{j=2}^n \int_\Omega \int_{t_{j-1}}^{t_j} \left| \int_{t_{j-2}}^{t_{j-1}} (\theta_n(x, t) - \theta_n(x, \tau)) d\tau \right|^2 dt dx \\ &\leq \frac{n}{T} \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \int_{t_{j-2}}^{t_{j-1}} \|\theta_n(t) - \theta_n(\tau)\|_H^2 d\tau dt \\ &\leq \frac{n}{T} \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \int_{t-t_{j-1}}^{t-t_{j-2}} \|\theta_n(t) - \theta_n(t-h)\|_H^2 dh dt \\ &\leq \frac{n}{T} \int_0^{2T/n} \left( \int_h^T \|\theta_n(t) - \theta_n(t-h)\|_H^2 dt \right) dh, \end{aligned}$$

where we have used the new variable  $h = t - \tau$ . The last integral tends to 0, because  $\theta_n$  converge strongly to  $\theta$ , which is mean continuous in  $L^2(Q_T)$ . This implies that

$$\bar{\theta}_n \rightarrow \theta \quad \text{strongly in } L^2(0, T; H),$$

and this allows us to pass to the limit in (4.4)–(4.6) as  $n \rightarrow \infty$ . Notice moreover that from estimate (4.30) we also deduce that there exists a function  $\zeta : (0, T) \rightarrow V$  such that

$$\log \theta_n \rightarrow \zeta \quad \text{weakly in } L^2(0, T; V).$$

Using the strong convergence of  $\theta_n$  and the maximal monotonicity of the extended log graph, we also deduce that  $\zeta = \log \theta \in L^2(0, T; V)$ , hence  $\theta > 0$  a.e.

## 4.5 Conclusion of the existence proof

Let us now introduce the limit problem obtained by passing to the limit as  $n \rightarrow \infty$  in (4.4)–(4.6) in the previous subsection.

**PROBLEM (P) $_{\varrho}$ .** For fixed  $T > 0$ , and  $\varrho > 0$ , find two functions  $\theta \in L^2(0, T; V)$  and  $\chi \in L^\infty(\Omega \times (0, T))^d$ ,  $\chi_t \in L^\infty(\Omega \times (0, T))^d$  such that, for  $t \in (0, T)$ , we have

$$(4.62) \quad \begin{aligned} & \langle (e(\theta(t), \chi(t)))_t, z \rangle + \int_{\Omega} k(\theta(t), \chi(t)) \nabla \theta(t) \cdot \nabla z \, dx + \int_{\partial\Omega} \gamma(\theta(t) - \theta_{\Gamma}) z \, dA \\ & = - \int_{\Omega} (\lambda'(\chi)(t) \chi_t(t) + \beta(\varphi(\chi(t)))_t - b[\chi](t) \chi_t(t)) z \, dx \quad \forall z \in V, \end{aligned}$$

$$(4.63) \quad \begin{aligned} & \tilde{\mu}_{\varrho}(\theta(t)) \chi_t(t) + (\beta + \theta(t)) \partial \varphi(\chi)(t) \ni -\lambda'(\chi)(t) - \theta(t) \sigma'(\chi)(t) \\ & - b[\chi](t) - e_{\chi}(\theta(t), \chi(t)) + \theta(t) s_{\chi}^{\varrho}(\theta(t), \chi(t)) \quad \text{a.e. in } \Omega, \end{aligned}$$

and for  $t = 0$  the functions  $e(\theta, \chi)$  and  $\chi$  satisfy the initial conditions

$$(4.64) \quad e(\theta, \chi)(0) = u_0, \quad \chi(0) = \chi_0.$$

We thus have proved the following result.

**Proposition 4.2.** *Let Hypothesis 2.1 hold true and let  $\varrho > 0$ ,  $\tilde{\mu}_{\varrho}$  and  $s_{\chi}^{\varrho}$  be defined as at the beginning of Section 4.1. Then PROBLEM (P) $_{\varrho}$  has a solution in  $(0, T)$ .*

In order to conclude the proof of Theorem 2.2, we only need to remove the truncation  $\varrho$ . This can be done using the Moser estimate (4.43).

This bound for  $\theta$ , indeed, allows us, for a suitable  $\varrho \geq 1$  satisfying

$$C^*(1 + \log \varrho)^{4+2N} \leq \varrho/2,$$

with, e.g.,  $C^* = \tilde{C}^2$ , to remove the truncation  $\varrho$  and to conclude that the solution to PROBLEM (P) $_{\varrho}$  is in fact a solution to PROBLEM (P).

Finally, let us note that we have now found a solution on  $(0, T)$  and we deduce that  $\chi$  is weakly continuous with values in  $\mathbf{V}$ ,  $e(\theta, \chi) \in C^0([0, T]; H)$ , and  $\theta$  is bounded uniformly in time in  $L^\infty(\Omega)$  due to the Moser estimate. Hence, we can continue the solution starting from time  $T$  and extend it on the whole time interval  $(0, \infty)$ .

This concludes the proof of Theorem 2.2. ■

## 5 Uniqueness

In this section, we prove Theorem 2.5. Hence, we assume Hypothesis 2.3. First let us note that the lower bound for  $\theta$  (2.18) directly follows by passing to the limit as  $n$  tends to  $\infty$  in (4.37). Now we are ready to proceed with the proof of uniqueness and finally we will prove the regularity result (2.21) under the further assumption (2.20).

**Uniqueness.** In what follows, we denote by  $R_0, R_1, R_2, \dots$  suitable constants that possibly depend on  $T$ , but not on the solutions. We start with the proof of uniqueness of solutions. Equation (2.2) is for (almost) all  $x \in \Omega$  of the form (2.10), with

$$\alpha(\theta) = \frac{\mu(\theta)}{\beta + \theta},$$

$$g = \ell[\theta, \chi] = -\frac{1}{\beta + \theta} (\lambda'(\chi) + \theta \sigma'(\chi) + b[\chi] + e_\chi(\theta, \chi) - \theta s_\chi(\theta, \chi)).$$

Within the range  $0 < \theta < \bar{\theta}$  and  $\chi \in \mathcal{D}_C(\varphi)$ ,  $|\chi_t| \leq C$ , of admissible values for the solutions, and, thanks to Hypothesis 2.1 (ii) (iii) (v) (vi), all nonlinearities in (2.1)–(2.2) are Lipschitz continuous. Using the notation from Theorem 2.5, we obtain, as a consequence of (2.17), for a.e.  $(x, t) \in Q_\infty$  the estimate

$$(5.1) \quad \int_0^t |\hat{\chi}_t(x, \tau)| d\tau + |\hat{\chi}(x, t)|$$

$$\leq R_0(T) \left( |\hat{\chi}_0(x)| + \int_0^t \left( |\hat{\theta}(x, \tau)| + |\hat{\chi}(x, \tau)| + \int_\Omega |\hat{\chi}(y, \tau)| dy \right) d\tau \right),$$

with some constant  $R_0(T)$ . Integrating over  $\Omega$ , and by Gronwall's argument, we obtain that

$$(5.2) \quad \int_\Omega |\hat{\chi}(y, t)| dy \leq R_1 \left( \int_\Omega |\hat{\chi}_0(y)| dy + \int_0^t \int_\Omega |\hat{\theta}(y, \tau)| dy d\tau \right),$$

and hence, we get

$$(5.3) \quad \int_0^t |\hat{\chi}_t(x, \tau)| d\tau + |\hat{\chi}(x, t)| \leq R_2 \left( |\hat{\chi}_0(x)| + \int_0^t |\hat{\theta}(x, \tau)| d\tau + \int_0^t \int_\Omega |\hat{\theta}(y, \tau)| dy d\tau \right)$$

for a.e.  $x \in \Omega$  and every  $t \in [0, T]$ . In particular,

$$(5.4) \quad \int_0^t \int_\Omega |\hat{\chi}_t(x, \tau)| dx d\tau \leq R_3 \left( \int_\Omega |\hat{\chi}_0(x)| dx + \int_0^t \int_\Omega |\hat{\theta}(x, \tau)| dx d\tau \right).$$

We now multiply (5.3) by  $|\hat{\chi}(x, t)|$  and integrate over  $\Omega$  to obtain for all  $t \in [0, T]$  that

$$(5.5) \quad \int_\Omega |\hat{\chi}(x, t)|^2 dx \leq R_4 \left( \|\hat{\chi}_0\|_{\mathbf{H}}^2 + \int_0^t \int_\Omega |\hat{\theta}(x, \tau)|^2 dx d\tau \right).$$

The crucial point is to exploit Eq. (2.1) properly. Notice first that we have

$$(5.6) \quad b[\chi] \chi_t(x, t) = 2B[\chi]_t(x, t) + 2 \int_\Omega \kappa(x, y) G'(\chi(x, t) - \chi(y, t)) \chi_t(y, t) dy.$$

We integrate the difference of the two equations (2.1), written for  $(\theta_1, \chi_1)$  and  $(\theta_2, \chi_2)$ , from 0 to  $t$ , rewriting the terms  $b[\chi_i](\chi_i)_t$  according to (5.6). Take  $z = K(\theta_1) - K(\theta_2)$  in the resulting equation, where  $K(u) = \int_0^u \bar{k}(s) \, ds$ ,  $u \in \mathbb{R}$ , and integrate it again over  $(0, t)$ . Using the lower bound for  $\theta$ , the Lipschitz continuity of all nonlinearities ( $\varphi$  is Lipschitz continuous on  $\mathcal{D}_C(\varphi)$  with constant  $C$ ), the properties of  $K$  (cf. Hypothesis 2.1(iv)), and denoting

$$\hat{\Theta}(x, t) = \int_0^t (K(\theta_1) - K(\theta_2))(x, \tau) \, d\tau,$$

using (4.37), we obtain for each  $t \in (0, T)$  that

$$(5.7) \quad \begin{aligned} & k_0 c_V(w) \int_0^t \int_{\Omega} |\hat{\theta}(x, \tau)|^2 \, dx \, d\tau + \frac{1}{2} \int_{\Omega} |\nabla \hat{\Theta}(x, t)|^2 \, dx \\ & \leq R_5 \left( \|\hat{\theta}_0\|_H^2 + \|\hat{\chi}_0\|_{\mathbf{H}}^2 + \int_0^t \int_{\Omega} |\hat{\chi}(x, \tau)|^2 \, dx \, d\tau \right. \\ & \quad \left. + \int_0^t \int_0^\xi \int_{\Omega} \int_{\Omega} \kappa(x, y) |\hat{\chi}_t(y, \tau)| |\hat{\theta}(x, t)| \, dx \, dy \, d\tau \, d\xi \right). \end{aligned}$$

The last term on the right-hand side of the above inequality can be estimated, using (5.4), by

$$\begin{aligned} & \int_0^t \int_{\Omega} \int_{\Omega} \kappa(x, y) |\hat{\chi}_t(y, \tau)| |\hat{\theta}(x, t)| \, dx \, dy \, d\tau \leq R_6 \int_{\Omega} |\hat{\theta}(x, t)| \, dx \int_0^t \int_{\Omega} |\hat{\chi}_t(y, \tau)| \, dy \, d\tau \\ & \leq R_7 \left( \int_{\Omega} |\hat{\theta}(x, t)|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\hat{\chi}_0(x)|^2 \, dx + \int_0^t \int_{\Omega} |\hat{\theta}(x, \tau)|^2 \, dx \, d\tau \right)^{1/2}. \end{aligned}$$

Combining the last two inequalities again with the Gronwall's lemma, we obtain for each  $t \in [0, T]$  the estimate

$$(5.8) \quad \begin{aligned} & \int_0^t \int_{\Omega} |\hat{\theta}(x, \tau)|^2 \, dx \, d\tau + \int_{\Omega} |\nabla \hat{\Theta}(x, t)|^2 \, dx \\ & \leq R_8 \left( \|\hat{\theta}_0\|_H^2 + \|\hat{\chi}_0\|_{\mathbf{H}}^2 + \int_0^t \int_{\Omega} |\hat{\chi}(x, \tau)|^2 \, dx \, d\tau \right). \end{aligned}$$

We now multiply (5.8) by  $2R_4$ , add the result to (5.5), and see that Gronwall's argument can be applied again to arrive at the final estimate

$$(5.9) \quad \int_{\Omega} |\hat{\chi}(x, t)|^2 \, dx + \int_0^t \int_{\Omega} |\hat{\theta}(x, \tau)|^2 \, dx \, d\tau \leq R_8 \left( \|\hat{\theta}_0\|_H^2 + \|\hat{\chi}_0\|_{\mathbf{H}}^2 \right).$$

**Regularity.** We prove now the regularity (2.21) for  $\theta$  under the further assumption (2.20). In order to do that, let us consider, instead of (4.4), the following approximated equation

$$(5.10) \quad \begin{aligned} & \int_{\Omega} \partial_t \left( \frac{1}{n} \theta_n(t) + e(\theta_n(t), \chi_n(t)) \right) z \, dx + \int_{\Omega} \bar{k}(\theta_n(t)) \nabla \theta_n(t) \cdot \nabla z \, dx \\ & = - \int_{\Omega} ((\lambda'(\chi_n)(t) + b[\chi_n](t)) \partial_t \chi_n(t) + \beta \partial_t (\varphi(\chi_n(t)))) z \, dx. \end{aligned}$$



Since (cf. Hypothesis 2.3(i)) the heat conductivity  $\bar{k}$  is independent of  $\chi$ , we can now test (5.10) by  $K(\theta_n)_t$ , where  $K(\theta_n) = \int_0^{\theta_n} \bar{k}(s) \, ds$  and, integrating over  $(0, t)$ , we obtain, using (4.37), the estimate

$$\begin{aligned} & k_0 \left( \frac{1}{n} + c_V(w) \right) \int_0^t \int_{\Omega} |(\theta_n)_t|^2 \, dx \, d\tau + \frac{1}{2} \|\nabla(K(\theta_n))(t)\|_H^2 \\ & \leq \frac{1}{2} \|\nabla(K(\theta_{0n}))\|_H^2 + Ck_1 \int_0^t \int_{\Omega} |(\theta_n)_t| \, dx \, d\tau. \end{aligned}$$

Here  $C$  is a bound in  $L^\infty(Q_T)$  for the term  $(\lambda(\chi_n) + b[\chi_n])\partial_t \chi_n + \beta \partial_t(\varphi(\chi_n))$  that we already obtained in Theorem 2.2, and  $k_0, k_1$  are the positive constants introduced in Hypothesis 2.1(iv). Using now assumption (2.20) and the conclusions of Theorem 2.2, we obtain the (independent of  $n$ ) bound

$$\|(\theta_n)_t\|_{L^2(0,T;H)} + \|K(\theta_n)\|_{L^\infty(0,T;V)} \leq C,$$

which leads immediately (due to Hypothesis 2.1(iv)) to the desired estimate (2.21).

With this, Theorem 2.5 is proved. ■

**Remark 5.1.** It seems to the authors that, in the case when  $k$  depends only on  $\theta$ , if we replace the boundary condition (1.3) with

$$(5.11) \quad k(\theta) \nabla \theta \cdot \mathbf{n} + \gamma(K(\theta) - K(\theta_\Gamma)) = 0 \quad \text{on } \partial\Omega,$$

where  $K$  is the primitive of  $k$  as in this section, then the uniqueness result holds for a general nonnegative  $\gamma$ . Moreover, the regularity estimates can be adapted in order to cover also this case: the details of this proof will be developed in a subsequent paper. However, let us point out that the regularity properties (2.21) remain true for more general  $\gamma$  also in the case of boundary condition (1.3): the argument relies on a suitable use of the Gronwall lemma.

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